On some generalisations of Brown’s conjecture

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Abstract

Let \( P \) be a complex polynomial of the form \( P(z) = z^{n-1} \prod_{k=1}^{n-1} (z - z_k) \), where \( |z_k| \geq 1, 1 \leq k \leq n - 1 \) then \( P'(z) \neq 0 \). If \( |z| < \frac{1}{n} \). In this paper, we present some interesting generalisations of this result.

Keywords: Critical points; Sendove’s Conjecture; Coincidence theorem of walsh.


1. Introduction and statement of the results

Let \( B(z, r) \) denote the open ball in \( C \) with centre \( z \) and radius \( r \) and \( \overline{B}(z, r) \) denote its closure. The Gauss Lucas Theorem states that every critical point of a complex polynomial \( P \) of degree at most \( n \) lies in the convex hull of its zeros. B. Sendove conjectured that if all the zeros of \( P \) lies in \( \overline{B}(0, 1) \) then for any zero \( w \) of \( P \) the disk \( \overline{B}(w, 1) \) contains at least one zero of \( P' \) see [4], problem 4.1. In connection with this conjecture Brown [3] posed the following problem.

Let \( Q_n \) denote the set of all complex polynomials of the form \( P(z) = z^{n-1} \prod_{k=1}^{n-1} (z - z_k) \), where \( |z_k| \geq 1, 1 \leq k \leq n - 1 \). Find the best constant \( C_n \) such that \( P'(z) \neq 0 \) in \( B(0, C_n) \) for all \( P \) in \( Q_n \). Brown conjectured that \( C_n = \frac{1}{n} \).

Recently, the conjecture was settled by Aziz and Zargar [2]. In fact by proving the following:

Theorem 1.1. For all \( P \) in \( Q_n \), \( P'(z) \) does not vanish if \( z \in \left( 0, \frac{1}{n} \right) \).

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Here, in this paper we shall present the following generalisation of Theorem 1.1.

**Theorem 1.2.** Let
\[ P(z) = z^m \prod_{j=1}^{n-m} (z - z_j) \]
be a polynomial of degree n, with \(|z_j| \geq 1, j = 1, 2, \ldots n - m\). Then for \(1 \leq r \leq m\), the polynomial \(P'(z)\), the \(r\)th derivative of \(P(z)\) does not vanish in
\[ 0 < |z| < \frac{m(m-1)(m-2)\cdots (m-r+1)}{n(n-1)(n-2)\cdots (n-r+1)} \]

**Remark 1.3.** Taking \(r = 1, m = 1\), we get Theorem A (Browns Conjecture).

The following result immediately follows from the proof of Theorem 1.2.

**Corollary 1.4.** Let
\[ P(z) = z^m \prod_{j=1}^{n-m} (z - z_j) \]
be a polynomial of degree \(n\), with \(|z_j| \geq 1, j = 1, 2, \ldots n - m\). Then the polynomial \(P''(z)\) does not vanish in
\[ 0 < |z| < \frac{m(m-1)}{n(n-1)} \]

Taking \(m = 2\) in Corollary 1.4, we get the following result.

**Corollary 1.5.** Let
\[ P(z) = z^2 \prod_{j=1}^{n-2} (z - z_j) \]
be a polynomial of degree \(n\), with \(|z_j| \geq 1, j = 1, 2, \ldots n - 2\). Then the polynomial \(P''(z)\) does not vanish in
\[ 0 < |z| < \frac{2}{n(n-1)} \]

2. **Lemmas**

For the proof of Theorem 1.2 we need the following lemmas. The first lemma is Walsh’s Coincidence Theorem [1, p 47] (see also [1]).

**Lemma 2.1.** If \(G(z_1, z_2, \ldots, z_n)\) is a symmetric \(n\)-linear form of total degree \(n\) in \((z_1, z_2, \ldots, z_n)\) and let \(C\) be a circular region containing the \(n\) points \(\alpha_1, \alpha_2, \ldots, \alpha_n\) then there exists at least one point \(\alpha\) in \(C\) such that
\[ G(\alpha_1, \alpha_2, \ldots, \alpha_n) = G(w_1, w_2, \ldots, w_n). \]

**Lemma 2.2.** If
\[ P(z) = z^m \prod_{k=1}^{n-m} (z - z_k) \]
be a polynomial of degree \(n\), with \(|z_k| \geq 1, 1 \leq k \leq n - m\). Then for the polynomial \(P'(z)\) does not vanish in
\[ 0 < |z| < \frac{m}{n} \]
Lemma 2.2 is due to Aziz and Zargar [2].

**Lemma 2.3.** If \( P(z) \) is a Polynomial of degree \( n \) such that \( P(z) \) does not vanish in \( |z| < 1 \), then the polynomial \( zP'(z) + 2P(z) \) does not vanish in \( |z| < \frac{2}{n + 2} \).

**Proof.** By hypothesis

\[
P(z) = \prod_{k=1}^{n} (z - z_k)
\]

is a polynomial of degree \( n \) having all its zeros in \( |z| \geq 1 \), so that \( |z_k| \geq 1, k = 1, 2, \ldots, n \). We prove all the zeros of

\[
H(z) = zP'(z) + 2P(z)
\]

lie in

\[
|z| \geq \frac{2}{n + 2}.
\]

To prove this let \( w \) be any zero of \( P(z) \) then

\[
H(w) = zP'(w) + 2P(w) = 0.
\]

Clearly \( H(z) \) is linear symmetric in the zeros \( z_1, z_2, \ldots, z_n \) of \( P(z) \). Therefore by Lemma 2.1, we can find atleast one point \( \beta \) with \( |\beta| \geq 1 \), such that

\[
P(z) = (z - \beta)^n.
\]

which gives

\[
H(w) = wnP'(w - \beta)^{n-1} + 2P(w - \beta) = 0
\]

which implies

\[
(w - \beta)^{n-1}[nw + 2(w - \beta)] = 0
\]

which gives,

\[
(w - \beta) = 0, \text{ or } nw + (w - \beta) = 0.
\]

If \( w - \beta = 0 \), then clearly \( |w| = |\beta| \geq 1 \). Now if,

\[
w + (w - \beta) = 0
\]

then

\[
w = \frac{2\beta}{n + 2}
\]

which gives,

\[
|w| = \frac{2}{n + 2}|\beta| \geq \frac{2}{n + 2}.
\]

Since \( w \) is any zero of

\[
H(z) = zP'(z) + 2P(z)
\]

therefore, it follows that

\[
zP'(z) + 2P(z)
\]

does not vanish in

\[
|z| < \frac{2}{n + 2},
\]

which completes the proof of Lemma 2.3. \( \square \)
3. Proof of Theorem

Proof. We have

\[ P(z) = z^m Q(z) \]

where

\[ Q(z) = \prod_{j=1}^{n-m} (z - z_j), \quad |z_j| \geq 1, j = 1, 2, ..., n - m. \]

So, it follows by Lemma 2.2 that \( P'(z) \) does not vanish in the disk

\[ 0 < |z| < \frac{m}{n}. \]

That is

\[ P'(z) = z^{m-1}(zQ'(z) + mzQ(z)) \]

where

\[ T(z) = (zQ'(z) + mzQ(z)) \]

does not vanish in

\[ 0 < |z| < \frac{m}{n}. \]

Replacing \( z \) by \( \frac{mz}{n} \), it follows that

\[ H(z) = P'\left(\frac{mz}{n}\right) \]

does not vanish in

\[ 0 < |z| < 1. \]

Now

\[ H(z) = P'(\frac{mz}{n}) = (\frac{m}{n})^{m-1} z^{m-1} T\left(\frac{mz}{n}\right). \]

Applying Lemma 2.2 to the polynomial \( H(z) \), it follows that \( H'(z) \) does not vanish in the disk

\[ 0 < |z| < \frac{m-1}{n-1}. \]

Replacing \( z \) by \( \frac{nz}{m} \), we get \( P''(z) \) does not vanish in

\[ 0 < |z| < \frac{m(m-1)}{n(n-1)}. \]

\( n \geq 2 \) which yields that,

\[ P''(z) = z^{m-1} T'(z) + (m-1) z^{m-2} T(z) \]

\[ = z^{m-1} (z T'(z) + (m-1) T(z)) \]

\[ = z^{m-1} R(z) \]
does not vanish in $0 < |z| < \frac{m(m-1)}{n(n-1)}$. Thus, it follows by Lemma 2.3 that

$$R(z) = (zT'(z) + (m-1)T(z))$$

does not vanish in $0 < |z| < \frac{m(m-1)}{n(n-1)}$. Replacing $z$ by $\frac{m(m-1)}{n(n-1)}z$, we have

$$R\left( \frac{m(m-1)}{n(n-1)}z \right) = (m-1)T\left( \frac{m(m-1)}{n(n-1)}z \right) + \left( \frac{m(m-1)}{n(n-1)}z \right) T\left( \frac{m(m-1)}{n(n-1)}z \right)$$

does not vanish in $0 < |z| < 1$. Therefore, it follows that

$$S(z) = P''\left( \frac{m(m-1)}{n(n-1)}z \right)$$

$$= \left( \frac{m(m-1)}{n(n-1)}z \right)^{m-1}z^{m-1}R\left( \frac{m(m-1)}{n(n-1)}z \right)$$

does not vanish in $0 < |z| < 1$. Applying Lemma 2.2 we get

$$S'(z) = P''\left( \frac{m(m-1)}{n(n-1)}z \right)$$

does not vanish in $0 < |z| < \frac{(m-2)}{(n-2)}$ and this completes the proof of Theorem 1.2.

References