Local higher derivations on $C^*$-algebras are higher derivations

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Abstract
Let $\mathfrak{A}$ be a Banach algebra. We say that a sequence $\{D_n\}_{n=0}^{\infty}$ of continuous operators form $\mathfrak{A}$ into $\mathfrak{A}$ is a local higher derivation if for each $a \in \mathfrak{A}$ there corresponds a continuous higher derivation $\{d_{a,n}\}_{n=0}^{\infty}$ such that $D_n(a) = d_{a,n}(a)$ for each non-negative integer $n$. We show that if $\mathfrak{A}$ is a $C^*$-algebra then each local higher derivation on $\mathfrak{A}$ is a higher derivation. We also prove that each local higher derivation on a $C^*$-algebra is automatically continuous.

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1. Introduction and preliminaries
Let $\mathfrak{A}$ be a Banach algebra. A continuous operator $\Delta : \mathfrak{A} \to \mathfrak{A}$ is called a local derivation if for each $a \in \mathfrak{A}$ there is a derivation $\delta_a : \mathfrak{A} \to \mathfrak{A}$ such that $\Delta(a) = \delta_a(a)$. A celebrated theorem of Johnson states that each local derivation on a $C^*$-algebra is a derivation. Taking idea from this concept, we introduce the notion of a local higher derivation and show that each local higher derivation on a $C^*$-algebra is indeed a higher derivation.

Though there is a continuity assumption in the definition of a local derivation, Johnson shows that we can omit this assumption when $\mathfrak{A}$ is a $C^*$-algebra. Similarly, we show that when the domain of a local higher derivation is a $C^*$-algebra, we can remove the continuity assumption from the definition of a local higher derivation and each local higher derivation on a $C^*$-algebra is automatically continuous even if not assumed a priori to be so.

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Proof. Let $a$ be an element of $\mathfrak{A}$. Since $\{D_n\}_{n=0}^{\infty}$ is a local higher derivation, there is a continuous higher derivation $\{d_{a,n}\}_{n=0}^{\infty}$ such that $D_n(a) = d_{a,n}(a)$ for each non-negative integer $n$.

We use induction on $n$. For $n = 0$ we have $D_1(a) = d_{a,1}(a) = d_{a,1}(D_0(a)) = d_{a,1}D_0(a)$. Thus if $\Delta_1 : \mathfrak{A} \to \mathfrak{A}$ is defined by $\Delta_1(a) = d_{a,1}$ for each $a \in \mathfrak{A}$, then $\Delta_1$ is a local derivation on $\mathfrak{A}$.

Now suppose that $\Delta_k$ is defined and is a local derivation for $k \leq n$. We can inductively assume that for each $a \in \mathfrak{A}$ and each $k \leq n$ there is a derivation $\delta_{a,k} : \mathfrak{A} \to \mathfrak{A}$, defined by $\delta_{a,k} = kd_{a,k} - \sum_{i=0}^{k-2} \delta_{a,i+1}d_{a,k-1-i}$, such that $\Delta_k(a) = \delta_{a,k}(a)$.

Putting $\Delta_{n+1} = (n+1)D_{n+1} - \sum_{k=0}^{n-1} \Delta_k D_{n-k}$, we show that the well-defined mapping $\Delta_{n+1}$ is a local derivation on $\mathfrak{A}$. To see this, suppose that $\delta_{a,n+1} = (n+1)d_{a,n+1} - \sum_{k=0}^{n-1} \delta_{a,k+1}d_{a,n-k}$. Clearly, $\Delta_{n+1}(a) = \delta_{a,n+1}(a)$. We show that $\delta_{a,n+1}$ is a derivation. For $x, y \in \mathfrak{A}$ we have

$$\delta_{a,n+1}(xy) = (n+1)d_{a,n+1}(xy) - \sum_{k=0}^{n-1} \delta_{a,k+1}d_{a,n-k}(xy)$$

$$= (n+1) \sum_{k=0}^{n} d_{a,k}(x)d_{a,n+1-k}(y) - \sum_{k=0}^{n-1} \delta_{a,k+1} \left( \sum_{\ell=0}^{n-k} d_{a,\ell}(x)d_{a,n-k-\ell}(y) \right).$$

Now we have

$$\delta_{a,n+1}(xy) = \sum_{k=0}^{n+1} (n+1)d_{a,k}(x)d_{a,n+1-k}(y) - \sum_{k=0}^{n-1} \delta_{a,k+1} \left( \sum_{\ell=0}^{n-k} d_{a,\ell}(x)d_{a,n-k-\ell}(y) \right)$$

$$= \sum_{k=0}^{n+1} (k + n + 1 - k)d_{a,k}(x)d_{a,n+1-k}(y) - \sum_{k=0}^{n-1} \delta_{a,k+1} \left( \sum_{\ell=0}^{n-k} d_{a,\ell}(x)d_{a,n-k-\ell}(y) \right).$$

Since $\delta_{a,1}, \ldots, \delta_{a,n}$ are derivations,

$$\delta_{a,n+1}(xy) = \sum_{k=0}^{n+1} kd_{a,k}(x)d_{a,n+1-k}(y) + \sum_{k=0}^{n+1} d_{a,k}(x)(n+1-k)d_{a,n+1-k}(y)$$

$$- \sum_{k=0}^{n+1} \sum_{\ell=0}^{n-k-1} \delta_{a,k+1}(d_{a,\ell}(x))d_{a,n-k-\ell}(y) + d_{a,\ell}(x)\delta_{a,k+1}(d_{a,n-k-\ell}(y)).$$

Writing

$$K = \sum_{k=0}^{n+1} kd_{a,k}(x)d_{a,n+1-k}(y) - \sum_{k=0}^{n+1} \sum_{\ell=0}^{n-k} \delta_{a,k+1}(d_{a,\ell}(x))d_{a,n-k-\ell}(y),$$

$$L = \sum_{k=0}^{n+1} d_{a,k}(x)(n+1-k)d_{a,n+1-k}(y) - \sum_{k=0}^{n+1} \sum_{\ell=0}^{n-k} d_{a,\ell}(x)\delta_{a,k+1}(d_{a,n-k-\ell}(y)).$$
we have $\delta_{a,n+1}(xy) = K + L$. Let us compute $K$ and $L$. In the summation $\sum_{k=0}^{n-1} \sum_{r=0}^{n-k} \delta_{a,k+1}(d_{a,r-k}(x))d_{a,n-r}(y)$ we have $0 \leq k + \ell \leq n$ and $k \neq n$. Thus if we put $r = k + \ell$ then we can write it as the form $\sum_{r=0}^{n-k} \sum_{k=0}^{n-r} \delta_{a,k+1}(d_{a,r-k}(x))d_{a,n-r}(y)$. Putting $\ell = r - k$ we indeed have

$$K = \sum_{k=0}^{n+1} kd_{a,k}(x)d_{a,n+1-k}(y) - \sum_{r=0}^{n} \sum_{0 \leq r, k \neq n} \delta_{a,k+1}(d_{a,r-k}(x))d_{a,n-r}(y)$$

$$= \sum_{k=0}^{n} kd_{a,k}(x)d_{a,n+1-k}(y) - \sum_{r=0}^{n} \sum_{k=0}^{r} \delta_{a,k+1}(d_{a,r-k}(x))d_{a,n-r}(y) - \sum_{k=0}^{n-1} \delta_{a,k+1}(d_{a,n-k}(x))y.$$

Putting $r + 1$ instead of $k$ in the first summation we have

$$K + \sum_{k=0}^{n-1} \delta_{a,k+1}(d_{a,n-k}(x))y$$

$$= \sum_{r=0}^{n} (r + 1)d_{a,r+1}(x)d_{a,n-r}(y) - \sum_{r=0}^{n} \sum_{k=0}^{r} \delta_{a,k+1}(d_{a,r-k}(x))d_{a,n-r}(y)$$

$$= \sum_{r=0}^{n-1} (r + 1)d_{a,r+1}(x) - \sum_{k=0}^{r} \delta_{a,k+1}(d_{a,r-k}(x)) \right] d_{a,n-r}(y) + (n + 1)d_{a,n+1}(x)y.$$

By our assumption $(r + 1)d_{a,r+1}(x) = \sum_{k=0}^{r} \delta_{a,k+1}(d_{a,r-k}(x))$ for $r = 0, \ldots, n - 1$. We can therefore deduce that

$$K = \left[(n + 1)d_{a,n+1}(x) - \sum_{k=0}^{n-1} \delta_{a,k+1}(d_{a,n-k}(x)) \right] y = \delta_{a,n+1}(x)y.$$

By a similar argument we have

$$L = x \left[(n + 1)d_{a,n+1}(y) - \sum_{k=0}^{n-1} \delta_{a,k+1}(d_{a,n-k}(y)) \right] = x\delta_{a,n+1}(y).$$

Thus

$$\delta_{a,n+1}(xy) = K + L = \delta_{a,n+1}(x)y + x\delta_{a,n+1}(y).$$

Whence $\delta_{a,n+1}$ is a derivation on $\mathfrak{A}$. $\square$

**Theorem 1.2.** Each local higher derivation $\{D_n\}_{n=0}^{\infty}$, with $D_0 = I$, from a $C^*$-algebra $\mathfrak{A}$ into itself is a higher derivation.

**Proof.** Proposition 1.1 implies the existence of sequence $\{\Delta_n\}_{n=0}^{\infty}$ of local derivations such that $(n + 1)D_{n+1} = \sum_{k=0}^{n} \Delta_{k+1}D_{n-k}$. The famous theorem of Johnson [8] now guarantees that $\Delta_n$ are derivations.

To see that $\{\Delta_n\}_{n=0}^{\infty}$ is a higher derivation, let $a, b \in \mathfrak{A}$ and $n$ be a non-negative integer. We use induction on $n$. For $n = 0$ we have $D_0(ab) = ab = D_0(a)D_0(b)$. Let us assume that

$$D_k(ab) = \sum_{i=0}^{k} D_i(a)D_{k-i}(b)$$

Thus

$$\Delta_{n+1}(ab) = \left(\sum_{i=0}^{n} \Delta_i(a) \Delta_{n-i}(b) \right) + \left(\sum_{i=0}^{n} \sum_{j=0}^{n-i} \Delta_i(a) \Delta_{n-i-j}(b) \right)$$

$$= \Delta_{n+1}(a)b + a\Delta_{n+1}(b) + \sum_{i=0}^{n} \sum_{j=0}^{n-i} \Delta_i(a) \Delta_{n-i-j}(b)$$

$$= \sum_{i=0}^{n} \Delta_i(a) \Delta_{n-i}(b) + \sum_{i=0}^{n} \sum_{j=0}^{n-i} \Delta_i(a) \Delta_{n-i-j}(b)$$

$$= \sum_{i=0}^{n+1} \Delta_i(a) \Delta_{n+1-i}(b)$$

whence

$$\Delta_n(ab) = ab.$$
for $k \leq n$. Thus we have
\[
(n + 1)D_{n+1}(ab) = \sum_{k=0}^{n} \Delta_{k+1}D_{n-k}(ab)
\]
\[
= \sum_{k=0}^{n} \Delta_{k+1} \sum_{i=0}^{n-k} D_i(a)D_{n-k-i}(b)
\]
\[
= \sum_{i=0}^{n} \left( \sum_{k=0}^{n-i} \Delta_{k+1}D_{n-k-i}(a) \right) D_i(b)
\]
\[
+ \sum_{i=0}^{n} D_i(a) \left( \sum_{k=0}^{n-i} \Delta_{k+1}D_{n-k-i}(b) \right).
\]

Using our assumption, we can write
\[
(n + 1)D_{n+1}(ab) = \sum_{i=0}^{n} (n - i + 1)D_{n-i+1}(a)D_i(b)
\]
\[
+ \sum_{i=0}^{n} D_i(a)(n - i + 1)D_{n-i+1}(b)
\]
\[
= \sum_{i=1}^{n+1} iD_i(a)D_{n+1-i}(b) + \sum_{i=0}^{n} (n + 1 - i)D_i(a)D_{n+1-i}(b)
\]
\[
= (n + 1) \sum_{k=0}^{n+1} D_k(a)D_{n+1-k}(b).
\]

\[\square\]

**Corollary 1.3.** Each local higher derivation \( \{D_n\}_{n=0}^{\infty} \), with \( D_0 = I \), from a \( C^* \)-algebra \( \mathfrak{A} \) into itself is automatically continuous.

**Proof.** We can inductively prove that each \( D_n \) is continuous. Clearly, \( D_0 = I \) is continuous. Let \( D_k \) be continuous for \( k \leq n \). A beautiful theorem of Sakai [15] states that each derivation on a \( C^* \)-algebra is automatically continuous. Thus \( \Delta_n \)'s of Proposition 1.1 are continuous. This implies that \( D_{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} \Delta_{k+1}D_{n-k} \) to be continuous as a linear combination of compositions of continuous operators. \[\square\]

**References**


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