A common fixed point theorem for weakly compatible maps satisfying common property (E.A.) and implicit relation in Intuitionistic fuzzy metric spaces

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Abstract

In this paper, employing the common property (E.A), we prove a common fixed theorem for weakly compatible mappings via an implicit relation in Intuitionistic fuzzy metric space. Our results generalize the results of S. Kumar [S. Kumar, Common fixed point theorems in Intuitionistic fuzzy metric spaces using property (E.A), J. Indian Math. Soc., 76 (1-4) (2009), 94–103] and C. Alaca et al. [C. Alaca, D. Turkoglu and C. Yildiz, Fixed points in Intuitionistic fuzzy metric spaces, Chaos Solitons and Fractals, 29 (2006), 1073–1078].

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1. Introduction

In 1986, Jungck [8] introduced the notion of compatible maps for a pair of self mappings. However, the study of common fixed points of non-compatible maps is also very interesting (see [16]). Aamri et al. [1] generalized the concept of non-compatibility by defining the notion of property (E.A) and in 2005, Liu et al. [3] defined common property (E.A) in metric spaces and proved common fixed point theorems under strict contractive conditions. Jungck et al. [9] initiated the study of weakly compatible maps in metric space and showed that every pair of compatible maps is weakly
compatible but reverse is not true. In the literature, many results have been proved for contraction maps satisfying property (E.A.) in different settings such as probabilistic metric spaces \[5\, 7\]; fuzzy metric spaces \[12\, 15\]; Intuitionistic fuzzy metric spaces \[11\, 19\]. In this paper, employing the common property (E.A), we prove a common fixed theorem for weakly compatible mappings via an implicit relation in Intuitionistic fuzzy metric space. Our results generalize the results of S. Kumar \[11\] and C. Alaca et al. \[2\].

2. Preliminaries and Definitions

The concepts of triangular norms (t-norms) and triangular conorms (t-conorms) are known as the axiomatic skeleton that we use are characterization fuzzy intersections and union respectively. These concepts were originally introduced by Menger \[14\] in study of statistical metric spaces.

Definition 2.1. \[18\] A binary operation \(* : [0, 1][0, 1] \rightarrow [0, 1]\) is continuous t-norm if * satisfies the following conditions:

(i) \(*) \) is commutative and associative;
(ii) \(* \) is continuous;
(iii) \(a * 1 = a\) for all \(a \in [0, 1]\)
(iv) \(a * b \leq c * d\) whenever \(a \leq c\) and \(b \leq d\) for all \(a, b, c, d \in [0, 1]\).

Definition 2.2. \[18\] A binary operation \(\Diamond : [0, 1][0, 1][0, 1] \rightarrow [0, 1]\) is continuous t-conorm if \(\Diamond \) satisfies the following conditions:

(i) \(\Diamond \) is commutative and associative;
(ii) \(\Diamond \) is continuous;
(iii) \(a\Diamond 0 = a\) for all \(a \in [0, 1]\)
(iv) \(a\Diamond b \leq c\Diamond d\) whenever \(a \leq c\) and \(b \leq d\) for all \(a, b, c, d \in [0, 1]\)

Alaca et al. \[2\] using the idea of Intuitionistic fuzzy sets, defined the notion of Intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil et al. \[10\] as:

Definition 2.3. \[2\] A 5-tuple \((X, M, N, *, \Diamond)\) is said to be an Intuitionistic fuzzy metric space if \(X\) is an arbitrary set, * is a continuous t-norm, \(\Diamond \) is a continuous t-conorm and \(M, N\) are fuzzy sets on \(X^2[0, \infty)\) satisfying the following conditions:

(i) \(M(x, y, t) + N(x, y, t) \leq 1\) for all \(x, y \in X\) and \(t > 0\);
(ii) \(M(x, y, 0) = 0\) for all \(x, y \in X\);
(iii) \(M(x, y, t) = 1\) for all \(x, y \in X\) and if and only if \(x = y\);
(iv) \(M(x, y, t) = M(y, x, t)\) for all \(x, y \in X\) and \(t > 0\);
(v) \(M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)\) for all \(x, y \in X\) and \(s, t > 0\);
(vi) for all \(x, y \in X\), \(M(x, y, .) : [0, \infty) \rightarrow [0, 1]\) is left continuous;
(vii) \(lim_{t \rightarrow \infty} M(x, y, t) = 1\) for all \(x, y \in X\) and \(t > 0\);
(viii) \(N(x, y, 0) = 1\) for all \(x, y \in X\);
(ix) \(N(x, y, t) = 0\) for all \(x, y \in X\) and \(t > 0\) if and only if \(x = y\);
(x) \(N(x, y, t) = N(y, x, t)\) for all \(x, y \in X\) and \(t > 0\);
(xi) \(N(x, y, t)\Diamond N(y, z, s) \geq N(x, z, t+s)\) for all \(x, y \in X\) and \(s, t > 0\);
(xii) for all \(x, y \in X\), \(N(x, y, .) : [0, \infty) \rightarrow [0, 1]\) is right continuous;
(xiii) $\lim_{t \to \infty} N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$.

Then $(M, N)$ is called an Intuitionistic fuzzy metric space on $X$. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between $x$ and $y$ w.r.t. $t$ respectively.

**Remark 2.4.** [2] Every fuzzy metric space $(X, M, *)$ is an Intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \Diamond)$ such that $t$-norm $*$ and $t$-conorm $\Diamond$ are associated as $x \Diamond y = 1 - ((1 - x) \ast (1 - y))$ for all $x, y \in X$.

**Remark 2.5.** [2] In Intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$, $M(x, y, .)$ is non-decreasing and $N(x, y, .)$ is non-increasing for all $x, y \in X$.

Alaca et al. [2] introduced the following notions:

**Definition 2.6.** Let $(X, M, N, *, \Diamond)$ be an Intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in $X$ is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$,

$$
\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1 	ext{ and } \lim_{n \to \infty} N(x_{n+p}, x_n, t) = 0;
$$

(b) a sequence $\{x_n\}$ in $X$ is said to be convergent to a point $x \in X$ if, for all $t > 0$,

$$
\lim_{n \to \infty} M(x_n, x, t) = 1 	ext{ and } \lim_{n \to \infty} N(x_n, x, t) = 0.
$$

**Definition 2.7.** [2] An Intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to be complete if and only if every Cauchy sequence in $X$ is convergent.

**Example 2.8.** [2] Let $X = \{ \frac{1}{n} : n \in N \} \cup \{0\}$ and let $*$ be the continuous $t$-norm and $\Diamond$ be the continuous $t$-conorm defined by and respectively, for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$ and $x, y \in X$, define $(M, N)$ by

$$
M(x, y, t) = \frac{t}{t + |x - y|} \text{ if } t > 0;
$$

$$
M(x, y, 0) = 0
$$

and

$$
N(x, y, t) = \frac{|x - y|}{t + |x - y|} \text{ if } t > 0;
$$

$$
N(x, y, 0) = 1
$$

Clearly, $(X, M, N, *, \Diamond)$ is complete Intuitionistic fuzzy metric space.

**Definition 2.9.** [1] A pair of self mappings $(T, S)$ of an Intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to satisfy the property $(E.A)$ if there exist a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} Tx_n = \lim_{n \to \infty} Sx_n = z$ for some $z \in X$.

**Example 2.10.** Let $X = [0, \infty)$. Consider $(X, M, N, *, \Diamond)$ be an Intuitionistic fuzzy metric space as in Example 2.8. Define $T, S : X \to X$ by $Tx = \frac{x}{3}$ and $Sx = \frac{2x}{3}$ for all $x \in X$. Clearly, for sequence $\{x_n\} = \{\frac{1}{n}\}$, $T$ and $S$ satisfies property $(E.A)$.

**Definition 2.11.** [2] Two pairs $(A, S)$ and $(B, T)$ of self mappings of an Intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ are said to satisfy the common property $(E.A)$ if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = z$ for some $z \in X$. 
Example 2.12. Let $X = [-1,1]$. Consider $(X, M, N, *, ∩)$ be an Intuitionistic fuzzy metric space as in Example 2.8. Define self mappings $A, B, S$ and $T$ on $X$ as $Ax = \frac{x}{2}, Bx = \frac{-x}{2}, Sx = x, Tx = -x$ for all $x \in X$. Then, with sequences $\{x_n\} = \{\frac{1}{n}\}$ and $\{y_n\} = \{\frac{-1}{n}\}$ in $X$, one can easily verify that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = 0$. Therefore, pairs $(A, S)$ and $(B, T)$ satisfies the common property $(E.A)$.

Definition 2.13. [9] A pair of self mappings $(T, S)$ of an Intuitionistic fuzzy metric space $(X, M, N, *, ∩)$ is said to be weakly compatible if they commute at coincidence points i.e. if $Tu = Su$ for some $u \in X$, then $TSu = STu$.

3. Main Results

Implicit relations play important role in establishing of common fixed point results. Let $M_4$ be the set of all real continuous functions $\psi, \phi : [0, 1]^4 \to R$, non-decreasing in the first argument and satisfying the following conditions:

(A) $\phi(u, 1, u, 1) \geq 0 \Rightarrow u \geq 1$,
(B) $\phi(u, 1, 1, u) \geq 0 \Rightarrow u \geq 1$,
(C) $\phi(u, u, 1, 1) \geq 0 \Rightarrow u \geq 1$,
(D) $\psi(u, 0, u, 0) \leq 0 \Rightarrow u \leq 0$,
(E) $\psi(u, 0, 0, u) \leq 0 \Rightarrow u \leq 0$,
(F) $\psi(u, u, 0, 0) \leq 0 \Rightarrow u \leq 0$

for all $u \geq 0$.

Example 3.1. Define $\psi, \phi : [0, 1]^4 \to R$ as $\phi(t_1, t_2, t_3, t_4) = 14t_1 - 12t_2 + 6t_3 - 8t_4$ and $\psi(t_1, t_2, t_3, t_4) = 12t_1 - 9t_2 + 8t_3 - 11t_4$. Clearly, and satisfies all conditions (A), (B), (C), (D), (E) and (F). Therefore, $\psi, \phi \in M_4$.

We begin with following observation:

Lemma 3.2. Let $\{A_i\}$, $S$ and $T$ be self mappings of an Intuitionistic fuzzy metric space $(X, M, N, *, ∩)$ satisfying the following:

(3.1) the pair $(A_0, T)$ satisfies the property $(E.A.)$;
(3.2) for any $x, y \in X$, and $\psi, \phi \in M_4$ and for all $t > 0$, there exists $k \in (0, 1)$ such that,
$\phi(M(A_i x, A_0 y, k t), M(S x, T y, t), M(S x, A_i x, t), M(T y, A_0 y, t)) \geq 0$;
$\psi(N(A_i x, A_0 y, k t), N(S x, T y, t), N(S x, A_i x, t), N(T y, A_0 y, t)) \leq 0$;
(3.3) $A_i(X) \subseteq T(X)$ or $A_0(X) \subseteq S(X)$.

Then the pairs $(A_i, S)$ and $(A_0, T)$ share the common $(E.A.)$ property.

Proof. As the pair $(A_0, T)$ satisfies property $(E.A.)$, then there exist a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} A_0 x_n = \lim_{n \to \infty} T x_n = z$ for some $z \in X$. Since $A_0(X) \subseteq S(X)$, hence for each $\{x_n\}$, there exist $\{y_n\}$ in $X$ such that $A_0 x_n = S y_n$.

Therefore, $\lim_{n \to \infty} A_0 x_n = \lim_{n \to \infty} S y_n = \lim_{n \to \infty} T x_n = z$. Now, we claim that $\lim_{n \to \infty} A_i y_n = z$. Suppose not, then applying inequality (3.2), we obtain
$\phi(M(A_i y_n, A_0 x_n, k t), M(S y_n, T x_n, t), M(S y_n, A_i y_n, t), M(T x_n, A_0 x_n, t)) \geq 0$;
$\psi(N(A_i y_n, A_0 x_n, k t), N(S y_n, T x_n, t), N(S y_n, A_i y_n, t), N(T x_n, A_0 x_n, t)) \leq 0$;

which on making $n \to \infty$ reduces to
Using (A) and (D), we get

\( M(\lim_{n \to \infty} A_i y_n, z, t), M(z, \lim_{n \to \infty} A_i y_n, t), M(z, z, t)) \geq 0; \)

As \( \phi \) and \( \psi \) is non-decreasing in the first argument, we have

\( \phi(M(\lim_{n \to \infty} A_i y_n, z, t), M(z, \lim_{n \to \infty} A_i y_n, t), M(z, z, t)) \geq 0; \)

\( \psi(N(\lim_{n \to \infty} A_i y_n, z, t), N(z, \lim_{n \to \infty} A_i y_n, t), N(z, z, t)) \leq 0; \)

Therefore, \( \lim_{n \to \infty} A_i y_n = z \). Hence, the pairs \( (A_i, S) \) and \( (A_0, T) \) share the common \( (E.A) \) property.

\( \square \)

**Theorem 3.3.** Let \( \{A_i\} \), \( S \) and \( T \) be self mappings of a Intuitionistic fuzzy metric space \( (X, M, N, *, \Diamond) \)
satisfying the conditions (3.2) and

(3.4) the pair \( (A_i, S) \) and \( (A_0, T) \) share the common property \((E.A)\);

(3.5) \( S(X) \) and \( T(X) \) are closed subsets of \( X \).

Then the pairs and have a point of coincidence each. Moreover, \( \{A_i\} \), \( S \) and \( T \) have a unique common
fixed point provided both the pairs \( (A_i, S) \) and \( (A_0, T) \) are weakly compatible.

**Proof.** In view of (3.4), there exist two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that

\( \lim_{n \to \infty} A_i y_n = \lim_{n \to \infty} A_i x_n = \lim_{n \to \infty} S x_n = \lim_{n \to \infty} T y_n = z \) for some \( z \in X \). Since \( S(X) \) is a closed subset of \( X \), therefore, there exists a point \( u \in X \) such that \( z = S u \). We claim that \( A_i u = z \). If \( A_i u \neq z \), then by

(3.2), take \( x = u, y = y_n \),

\( \phi(M(A_i u, A_0 y_n, kt), M(S u, T y_n, t), M(S u, A_i u, t), M(T y_n, A_0 y_n, t)) \geq 0; \)

on making \( n \to \infty \), we get

\( \phi(M(A_i u, z, kt), M(z, z, t), M(z, A_i u, t), M(z, z, t)) \geq 0; \)

\( \psi(M(A_i u, z, kt), 1, M(z, A_i u, t), 1) \geq 0; \)

and

\( \psi(N(A_i u, A_0 y_n, kt), N(S u, T y_n, t), N(S u, A_i u, t), N(T y_n, A_0 y_n, t)) \leq 0; \)

on making \( n \to \infty \), we get

\( \psi(N(A_i u, z, kt), N(z, z, t), N(z, A_i u, t), N(z, z, t)) \leq 0; \)

\( \psi(N(A_i u, z, kt), 0, N(z, A_i u, t), 0) \leq 0; \)

As \( \phi \) and \( \psi \) is non-decreasing in the first argument, we have

\( \phi(M(A_i u, z, t), 1, M(z, A_i u, t), 1) \geq 0; \)

and

\( \psi(N(A_i u, z, t), 0, N(z, A_i u, t), 0) \leq 0; \)

Using (A) and (D), we get \( M(A_i u, z, t) \geq 1 \) and \( N(A_i u, z, t) \leq 0. \)

Hence \( M(A_i u, z, t) = 1 \) and \( N(A_i u, z, t) = 0. \)

Therefore, \( A_i u = z = S u \) which shows that \( u \) is a coincidence point of the pair \( (A_i, S) \).

Since \( T(X) \) is also a closed subset of \( X \), therefore \( \lim_{n \to \infty} T y_n = z \) in \( T(X) \) and hence there exists
\( v \in X \) such that \( T v = z = A_i u = S u. \)

Now, we show that \( A_0 v = z \). If not, then by using inequality (3.2), take \( x = u, y = v \), we have

\( \phi(M(A_i u, A_0 v, kt), M(S u, T v, t), M(S u, A_i u, t), M(T v, A_0 v, t)) \geq 0; \)

\( \phi(M(z, A_0 v, kt), 1, 1, M(z, A_0 v, t)) \geq 0; \)

and

\( \psi(N(A_i u, A_0 v, kt), N(S u, T v, t), N(S u, A_i u, t), N(T v, A_0 v, t)) \leq 0; \)

\( \psi(N(z, A_0 v, kt), 0, 0, N(z, A_0 v, t)) \leq 0; \)

As \( \phi \) and \( \psi \) is non-decreasing in the first argument, we have
\[ \phi(M(z, A_0v, kt), 1, 1, M(z, A_0v, t)) \geq 0; \]
and
\[ \psi(N(z, A_0v, kt), 0, 0, N(z, A_0v, t)) \leq 0; \]
Using (B) and (E), we get \( M(z, A_0v, kt) \geq 1 \) and \( N(z, A_0v, kt) \leq 0. \)
Hence \( M(z, A_0v, kt) = 1 \) and \( N(z, A_0v, t) = 0. \) Therefore, \( A_0v = z = Tv \) which shows that \( v \) is a coincidence point of the pair \((A_0, T)\).
Since the pairs \((A_i, S)\) and \((A_0, T)\) are weakly compatible and \( A_iu = Su, A_0v = Tv \), therefore, \( A_iz = A_iSu = SA_iu = Sz, A_0z = A_0Tv = TA_0v = Tz. \) If \( A_iz \neq z \), then by using inequality (3.2), we have
\[ \phi(M(A_iz, A_0v, kt), M(Sz, Tv, t), M(Sz, A_iz, t), M(Tv, A_0v, t)) \geq 0; \]
\[ \phi(M(A_iz, z, kt), M(A_iz, z, t), M(A_iz, A_iz, t), M(z, z, t)) \geq 0; \]
\[ \phi(M(A_iz, z, kt), M(A_iz, z, t), 1, 1) \geq 0; \]
and
\[ \psi(N(A_iz, A_0v, kt), N(Sz, Tv, t), N(Sz, A_iz, t), N(Tv, A_0v, t)) \leq 0; \]
\[ \psi(N(A_iz, z, kt), N(A_iz, z, t), N(A_iz, A_iz, t), N(z, z, t)) \leq 0; \]
\[ \psi(N(A_iz, z, kt), N(A_iz, z, t), 0, 0) \leq 0; \]
As \( \phi \) and \( \psi \) is non-decreasing in the first argument, we have
\[ \phi(M(A_iz, z, t), M(A_iz, z, t), 1, 1) \geq 0; \]
and
\[ \psi(N(A_iz, z, kt), N(A_iz, z, t), 0, 0) \leq 0; \]
Using (C) and (F), we get
\[ M(A_iz, z, t) \geq 1 \] and \( N(A_iz, z, t) \leq 0. \)
Hence \( M(A_iz, z, t) = 1 \) and \( N(A_iz, z, t) = 0. \)
Therefore, \( A_iz = z = Sz. \)
Similarly, one can prove that \( A_0z = Tz = z. \) Hence \( A_0z = A_iz = Sz = Tz \) and \( z \) is common fixed point of \( A_i, A_0, S \) and \( T. \)
Uniqueness: Let \( z \) and \( w \) be two common fixed points of \( A_i, A_0, S \) and \( T. \) If \( z \neq w \), then by using inequality (3.2), we have
\[ \phi(M(A_iz, A_0w, kt), M(Sz, Tw, t), M(Sz, A_iz, t), M(Tw, A_0w, t)) \geq 0; \]
\[ \phi(M(z, w, kt), M(z, w, t), M(z, z, t), M(w, w, t)) \geq 0; \]
\[ \phi(M(z, w, t), M(z, w, t), M(z, t), M(w, w, t)) \geq 0; \]
\[ \phi(M(z, w, t), M(z, w, t), 1, 1) \geq 0; \]
and
\[ \psi(N(A_iz, A_0w, kt), N(Sz, Tw, t), N(Sz, A_iz, t), N(Tw, A_0w, t)) \leq 0; \]
\[ \psi(N(z, w, kt), N(z, w, t), N(z, z, t), N(w, w, t)) \leq 0; \]
\[ \psi(N(z, w, t), N(z, w, t), N(z, t), N(w, w, t)) \leq 0; \]
\[ \psi(N(z, w, t), N(z, w, t), 0, 0) \leq 0; \]
Using (C) and (F), we have
\[ M(z, w, t) \geq 1 \] and \( N(z, w, t) \leq 0. \)
Hence, \( M(z, w, t) = 1 \) and \( N(z, w, t) = 0. \)
Therefore, \( z = w. \)

By choosing \( A_i, A_0, S \) and \( T \) suitably, one can derive corollaries involving two or more mappings. As a sample, we deduce the following natural result for a pair of self mappings by setting \( A_0 = A_i \) and \( T = S \) in above theorem:

**Corollary 3.4.** Let \( A_i \) and \( S \) be self mappings of an Intuitionistic fuzzy metric space \((X, M, N, *, \diamond)\) satisfying the following:
The following example illustrates Theorem 3.3.

Example 3.5. Let \((X, M, N, *, \Diamond)\) be an Intuitionistic fuzzy metric space as in Example 2.8 where \(X = [0, 2]\) and define \(\psi, \phi : [0, 1]^4 \to R\) as \(\phi(t_1, t_2, t_3, t_4) = 14t_1 - 12t_2 + 6t_3 - 8t_4\) and \(\psi(t_1, t_2, t_3, t_4) = 12t_1 - 9t_2 + 8t_3 - 11t_4\). Clearly, all conditions \((A), (B), (C), (D), (E)\) and \((F)\) hold. Therefore \(\psi, \phi \in M_4\).

Define \(A_i, A_0, S\) and \(T\) by
\[
A_i x = A_0 x = 1, \\
S x = \begin{cases} 1 & \text{if } x \in Q, S x = \frac{2}{3} & \text{otherwise} \\
T x = \begin{cases} 1 & \text{if } x \in Q, T x = \frac{1}{5} & \text{otherwise}
\end{cases}
\]
for all \(x, y \in X = [0, 2]\) and \(t > 0\). Then with sequences \(\{x_n = \frac{1}{n}\}\) and \(\{y_n = \frac{-1}{n}\}\) in \(X\), we have \(\lim_{n \to \infty} A_0 x_n = \lim_{n \to \infty} A_1 y_n = \lim_{n \to \infty} S y_n = \lim_{n \to \infty} T x_n = 1 \in X\) which shows that pairs \((A_i, S)\) and \((A_0, T)\) share the common property \((E.A)\). By a routine calculation, one can verify the condition (3.2). Thus, all the conditions of Theorem 3.3 are satisfied and \(x = 1\) is the unique common fixed point of \(A_i, A_0, S\) and \(T\).

References

