Some Inequalities Involving Lower Bounds of Operators on Weighted Sequence Spaces by a Matrix Norm

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Abstract

Let $A = (a_{n,k})_{n,k \geq 1}$ and $B = (b_{n,k})_{n,k \geq 1}$ be two non-negative matrices. Denote by $L_{v,p,q,B}(A)$, the supremum of those $L$, satisfying the following inequality:

$$\|Ax\|_{v,B(q)} \geq L \|x\|_{v,B(p)},$$

where $x \geq 0$ and $x \in l_p(v,B)$ and also $v = (v_n)_{n=1}^{\infty}$ is an increasing, non-negative sequence of real numbers. In this paper, we obtain a Hardy-type formula for $L_{v,p,q,B}(H_\mu)$, where $H_\mu$ is the Hausdorff matrix and $0 < q \leq p \leq 1$. Also for the case $p = 1$, we obtain $\|A\|_{w,B(1)}$, and for the case $p \geq 1$, we obtain $L_{w,B(p)}(A)$.

Keywords: Lower Bound, Weighted Block Sequence Space, Hausdorff Matrices, Euler Matrices, Cesàro Matrices, Matrix Norm.

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1. Introduction

Suppose that $v = (v_n)_{n=1}^{\infty}$ is an increasing, non-negative sequence of real numbers with $v_1 = v_2 = 1$ and $\sum_{n=1}^{\infty} \frac{v_n}{n} = \infty$. For $p \in R \backslash \{0\}$, let $l_p(v)$ denotes the space of all real sequences $x = \{x_k\}_{k=1}^{\infty}$, such that

$$\|x\|_{v,p} := \left(\sum_{k=1}^{\infty} v_k |x_k|^p \right)^{\frac{1}{p}} < \infty.$$
Lashkaripour and Foroutannia in [10], defined the weighted block sequence space as follows. Assume that $F = (F_n)$ is a partition of positive integers where each $F_n$ is a finite interval of $\mathbb{N}$ and

$$\max F_n < \min F_{n+1} \quad (n = 1, 2, \ldots).$$

The weighted block sequence space $l_p(v, F)$ is defined as

$$l_p(v, F) := \left\{ x = (x_n) : \sum_{n=1}^{\infty} v_n |x_n| < x, F_n > |p < \infty \right\},$$

where, $< x, F_n >= \sum_{j \in F_n} x_j$. The norm on $l_p(v, F)$ is denoted by $\|:\|_{p,v,F}$ and is defined by

$$\|x\|_{p,v,F} := \left( \sum_{n=1}^{\infty} v_n \left( \sum_{j \in F_n} |x_j| \right)^p \right)^{\frac{1}{p}}.$$

(1.1)

Note that with the above-mentioned definition $l_p(v, F)$ is not a norm sequence space. Indeed, one may consider $x = (1, -1, 0, 0, \ldots)$, $F_1 = \{1, 2\}$, $F_2 = \{3, 4\}$, ... and $v_n = 1$ then, $\|x\|_{p,v,F} = 0$ whereas $x \neq 0$.

We reform definition 1.1 as

$$l_p(v, F) := \left\{ x = (x_n) : \sum_{n=1}^{\infty} v_n \left( \sum_{j \in F_n} |x_j| \right)^p < \infty \right\},$$

and

$$\|x\|_{p,v,F} := \left( \sum_{n=1}^{\infty} v_n \left( \sum_{j \in F_n} |x_j| \right)^p \right)^{\frac{1}{p}}.$$

(1.2)

Of course, for non-negative sequences two definitions are coincide.

G. Bennett in [3] by a matrix $A$ with non-negative entries and $p > 0$, defined the sequence space

$$l_A(p) = \left\{ x = (x_n) : \sum_{n} \left( \sum_{k} a_{n,k} |x_k| \right)^p < \infty \right\}.$$

For $p \geq 1$ with the norm

$$\|x\|_{A(p)} = \left( \sum_{n} \left( \sum_{k} a_{n,k} |x_k| \right)^p \right)^{\frac{1}{p}},$$

(1.3)

$l_A(p)$ is a norm sequence space.

By a partition $F = (F_n)$, we correspond a matrix $A = (a_{n,k})$ such that $a_{n,k} = 1$, for $k \in F_n$ and $a_{n,k} = 0$, otherwise. One may easily verifies that

$$\|x\|_{v,A(p)} = \|x\|_{p,v,F},$$

where,

$$\|x\|_{v,A(p)} = \left( \sum_{n} v_n \left( \sum_{k} a_{n,k} |x_k| \right)^p \right)^{\frac{1}{p}}.$$

(1.4)
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For any partition, the corresponding matrix is a quasi-summability matrix, which is an upper triangular matrix which has column-sums 1.

For a certain \( I_n \) such as \( I_n = \{ n \} \), \( I = (I_n) \), is a partition of positive integers, \( l_p(v, I) = l_{I(p)}(v) = l_p(v) \), and \( \|x\|_{v, p, I} = \|x\|_{v, I(p)} = \|x\|_{v, p} \).

We write \( x \geq 0 \) if \( x_k \geq 0 \) for all \( k \). For \( p, q \in \mathbb{R} \setminus \{0\} \), the lower bound involved here is the number \( L_{v, p, q, B}(A) \), which is defined as the supremum of those \( L \) obeying the following inequality:

\[
\|Ax\|_{v, B(q)} \geq L \|x\|_{v, B(p)},
\]

where \( x \geq 0, x \in l_{B(p)}(v) \) and \( A = (a_{n,k})_{n,k \geq 1} \) is a non-negative matrix operator from \( l_{B(p)}(v) \) into \( l_{B(q)}(v) \). Also \( B = (b_{n,k})_{n,k \geq 1} \) is a non-negative matrix.

In this study, \( d\mu \) is a Borel probability measure on \([0, 1]\) and \( H_{\mu} = (h_{n,k})_{n,k \geq 0} \) is the Hausdorff matrix associated with \( d\mu \), defined by

\[
h_{n,k} = \begin{cases} 
\binom{n}{k} \int_0^1 \theta^k (1 - \theta)^{n-k} d\mu(\theta) & (n \geq k), \\
0 & (n < k).
\end{cases}
\]

Clearly, \( h_{n,k} = \binom{n}{k} \Delta^{n-k} \mu_k \) for \( n \geq k \geq 0 \), where

\[
\mu_k = \int_0^1 \theta^k d\mu(\theta) \quad (k = 0, 1, \ldots),
\]

and \( \Delta \mu_k = \mu_k - \mu_{k+1} \).

The Hausdorff matrix contains some famous classes of matrices. These classes are as follows:

i) Choosing \( d\mu(\theta) = \alpha (1 - \theta)^{\alpha-1} d\theta \) gives the Cesàro matrix of order \( \alpha \);

ii) Choosing \( d\mu(\theta) = \text{point evaluation at } \theta = \alpha \) gives the Euler matrix of order \( \alpha \);

iii) Choosing \( d\mu(\theta) = |\log \theta|^{\alpha-1}/\Gamma(\alpha) d\theta \) gives the Hölder matrix of order \( \alpha \);

iv) Choosing \( d\mu(\theta) = \alpha \theta^{\alpha-1} d\theta \) gives the Gamma matrix of order \( \alpha \).

The Cesàro, Hölder and Gamma matrices have non-negative entries whenever \( \alpha > 0 \), and also the Euler matrix has non-negative entries when \( 0 \leq \alpha \leq 1 \).

The study of \( L_{p,q}(A) \) goes back to the work of Copson. In \([7]\)(see also\([8]\) Theorem 344) he proved that \( L_{p,q}(C^1(1)) = p \) for \( 0 < p \leq 1 \), where \( C^1(1) = (a_{n,k})_{n,k \geq 0} \) is the Cesàro matrix defined by

\[
a_{n,k} = \begin{cases} 
\frac{1}{n+1} & (0 \leq k \leq n), \\
0 & (k > n).
\end{cases}
\]

These results extended by Bennett in many ways (cf, \([1],[2],[3],[4]\)). In particular, in \([3]\), Theorem 7.18), he proved that

\[
L_{p,p}(H_{\mu}) = \int_0^1 \theta^{-\frac{1}{p}} d\mu(\theta) \quad (0 < p \leq 1),
\]

where \( \frac{1}{p} + \frac{1}{p} = 1 \). According to \([3]\), Proposition 7.9], \(1.5\) also gives

\[
L_{p,p}(H_{\mu}) = \int_0^1 \theta^{-\frac{1}{p}} d\mu(\theta) \quad (-\infty < p < 0).
\]

(1.6)
This is a Hardy-type formula (cf. [3], Eq. (1-8)). The difference between them is that (1.6) is about $L_{p,p}(H_\mu)$, while Eq. (1-8) in [3] is about $\|H_\mu\|_{p,p}$.

Chen and Wang in [5] proved that $L_{p,p}(H_\mu) = \mu(\{1\})$ and $L_{p,p}(H_\mu^t) = \left( (\mu(\{0\})^q + (\mu(\{1\})^q \right)^{\frac{1}{q}}$, where $1 < q \leq p \leq \infty$. The case $0 < q < 1 \leq p \leq \infty$ is also examined there. Also in [6], they computed the exact values of $L_{p,p}(H_\mu)$ ($0 < p < 1$) and $L_{p,p}(H_\mu^t)$ ($-\infty < p < 0$) as follows:

$$L_{p,q}(H_\mu) \geq \int_{(0,1]} \theta^{-\frac{1}{q}} d\mu(\theta) \quad (0 < q \leq p \leq 1) \quad (1.7)$$

and

$$L_{p,q}(H_\mu^t) \geq \int_{(0,1]} \theta^{-\frac{1}{p^*}} d\mu(\theta) \quad (-\infty < q < p < 1).$$

Lashkaripour and G. talebi in [11] proved the following theorem.

Theorem 1.1. (11, Theorem 2.4.) For the Hausdorff matrix $H_\mu$ and partition $F = (F_n)$ we have

$$L_{v,p,q,F}(H_\mu) \geq \int_{(0,1]} \theta^{-\frac{1}{q}} d\mu(\theta) \quad (0 < q \leq p \leq 1). \quad (1.8)$$

Moreover, the following statements are true:

i) For $p = q = 1$, (1.8) is an equality.

ii) For $0 < q < p \leq 1$ and $F_n = I_n$, (1.8) is an equality if and only if $\mu(\{0\}) + \mu(\{1\}) = 1$ or the right-hand side of (1.8) is infinity.

In this paper, we improve and generalize the above-mentioned theorem. Also, we generalize some theorems on $l_p(w,F)$, which have proved by Lashkaripour and Foroutannia to the space $l_w,B(p)$.

2. New results

Proposition 2.1. Suppose that $0 < p < 1$, and let $A = (a_{n,k})$ and $B = (b_{n,k})$ be two matrices with non-negative entries. If we take

$$\sup_{n \geq 1} \sum_{k=1}^{\infty} a_{n,k} = R_A, \quad \inf_{k \geq 1} \sum_{n=1}^{\infty} a_{n,k} = C_A$$

and

$$\sup_{i \geq 1} \sum_{j=1}^{\infty} b_{i,j} = R_B, \quad \inf_{j \geq 1} \sum_{i=1}^{\infty} b_{i,j} = C_B$$

then for $x \geq 0$, we have

$$\|Ax\|_{v,B(p)} \geq \frac{L}{\|x\|_{v,p}}$$

with

$$L \geq (C_B C_A)^{\frac{1}{p}} (R_A R_B)^{\frac{1}{p^*}}.$$
\textbf{Proof.} By taking \( y_j = (Ax)_j = \sum_{k=1}^{\infty} a_{j,k} x_k \) and applying Hölder's inequality, we have
\[
\sum_{k=1}^{\infty} a_{n,k} v_k y_k^p = \sum_{k=1}^{\infty} a_{n,k}^{1-p} (a_{n,k} v_k^{\frac{1}{p}} y_k)^p \leq \left( \sum_{k=1}^{\infty} a_{n,k} \right)^{1-p} \left( \sum_{k=1}^{\infty} a_{n,k} v_k^{\frac{1}{p}} y_k \right)^p \leq R_A^{1-p} \left( \sum_{k=1}^{\infty} a_{n,k}^{\frac{1}{p}} y_k \right)^p.
\]

By similar way
\[
\sum_{j=1}^{\infty} b_{i,j} v_j y_j^p \leq R_B^{-1-p} \left( \sum_{j=1}^{\infty} b_{i,j}^{1-p} y_j \right)^p.
\]
Since \( v \) is increasing, we have
\[
R_A^{1-p} R_B^{1-p} \|Ax\|_{v,B(p)}^{p} = R_A^{1-p} R_B^{1-p} \left( \sum_{i=1}^{\infty} v_i \left( \sum_{j=1}^{\infty} b_{i,j} y_j \right)^p \right) \geq R_A^{1-p} R_B^{1-p} \left( \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} b_{i,j} v_j^{\frac{1}{p}} y_j \right)^p \right) \geq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} b_{i,j} \left( \sum_{k=1}^{\infty} a_{j,k} v_k x_k^p \right) = \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} a_{j,k} \left( \sum_{i=1}^{\infty} b_{i,j} \right) \right) v_k x_k^p \geq C_B C_A \sum_{k=1}^{\infty} v_k x_k^p,
\]
and this leads us to the desired inequality. \( \square \)

\textbf{Remark 2.2.} By taking \( B = I \) and \( v_n = 1 \) in above statement we obtain the following conclusion due Bennett ([3] Proposition 7.4.):

Fix \( p, 0 < p < 1 \), and suppose that \( A \) is a matrix with non-negative entries. If \( \sup_n \sum_{k=1}^{\infty} a_{n,k} = R \) and \( \inf_k \sum_{n=1}^{\infty} a_{n,k} = C \), then \( L_{p,q}(A) \geq R^{\frac{1}{p}} C^{\frac{1}{q}} \).

For \( \alpha \geq 0 \), let \( E(\alpha) = (e_{n,k}(\alpha))_{n,k \geq 1} \) denotes the Euler matrix, defined by
\[
e_{n,k}(\alpha) = \begin{cases} \binom{n-1}{k-1} \alpha^k (1 - \alpha)^{n-k} & (n \geq k), \\ 0 & (n < k). \end{cases}
\]
(cf. [6]). For $\Omega \subset (0, 1]$, we have

$$\int_{\Omega} e_{n,k}(\theta) d\mu(\theta) = \mu(\Omega) \times \int_{0}^{1} e_{n,k}(\theta) d\lambda(\theta),$$

where, $d\lambda = \frac{\mu(\Omega)}{\mu([0, 1])} d\mu$ is a Borel probability measure on $[0, 1]$ with $\lambda(\{0\}) = 0$. Hence the second part of ([3], Proposition 19.2) can be generalized in the following way.

Proposition 2.3. Suppose that $0 < p \leq 1, \Omega \subseteq [0, 1]$ and $d\mu$ is any Borel probability measure on $[0, 1]$. If $\mu(\{0\}) = 0$ or $\Omega \subset (0, 1]$, then the sequence $\|\left\{ \int_{\Omega} e_{n,k}(\theta) d\mu(\theta) \right\}_{n=k}^{\infty}$ increase with respect to $k$.

Proposition 2.4. Suppose that $0 < p \leq 1$ and $B$ is a matrix with non-negative entries, then for $0 < \alpha \leq 1$, we have

$$L_{v,B}(E(\alpha)) \geq C_{B}^{\frac{1}{p}} R_{B}^{\frac{1}{q}} \alpha^{-\frac{1}{q}}.$$

Proof. One may easily verifies that $\sum_{k=1}^{\infty} e_{n,k}(\alpha) = 1(n \geq 1)$ and $\sum_{n=1}^{\infty} e_{n,k}(\alpha) = \alpha^{-1}(k \geq 1)$. Applying Proposition 2.1 to case that $R_{A} = 1$ and $C_{A} = \alpha^{-1},$for $0 < p < 1$, we deduce that

$$L_{v,B}(E(\alpha)) \geq C_{B}^{\frac{1}{p}} R_{B}^{\frac{1}{q}} \alpha^{-\frac{1}{q}}.$$

For $p = 1$, by the Fubini’s theorem and monotonicity of $(v_{n})$, we deduce that

$$\|E(\alpha)x\|_{v,B(1)} = \sum_{i=1}^{\infty} v_{i} \left( \sum_{j=1}^{\infty} b_{i,j} y_{j} \right)$$

$$\geq \sum_{j=1}^{\infty} v_{j} y_{j} \left( \sum_{i=1}^{\infty} b_{i,j} \right)$$

$$\geq C_{B} \sum_{j=1}^{\infty} v_{j} \left( \sum_{k=1}^{\infty} e_{j,k}(\alpha) x_{k} \right)$$

$$\geq C_{B} \sum_{k=1}^{\infty} v_{k} x_{k} \left( \sum_{j=1}^{\infty} e_{j,k}(\alpha) \right)$$

$$= C_{B} \alpha^{-1} \|x\|_{v,1},$$

which gives the desired inequality. This completes the proof.

Theorem 2.5. By the previous assumptions on $B$ and $v$, we have

$$L_{v,p,q,B}(H_{\mu}) \geq C_{B}^{\frac{1}{p}} R_{B}^{\frac{1}{q}} \int_{[0,1]} \theta^{-\frac{1}{q}} d\mu(\theta) \quad (0 < q \leq p \leq 1).$$

(2.1)

Moreover, the following statements are true:

(i) For $p = q = 1$, (2.1) is an equality, if $B$ is a quasi-summability matrix.

(ii) For $0 < q < p \leq 1$ or $B = I$(the identity matrix), (2.1) is an equality if and only if $\mu(\{0\}) + \mu(\{1\}) = 1$ or the right-hand side of 2.1 is infinity.
**Proof.** Suppose that $x \geq 0$ with $\|x\|_{v,B(p)} = 1$, then $\|x\|_{v,B(q)} \geq \|x\|_{v,B(p)} = 1$. Applying Minkowski’s inequality and Proposition 2.3, we have

$$
\|H_{\mu}(x)\|_{v,B(q)} = \left( \sum_{n=1}^{\infty} v_n \left( \sum_{k=1}^{\infty} b_{n,k} (H_{\mu}(x)_k)^q \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}
$$

$$
= \left( \sum_{n=1}^{\infty} v_n \left( \sum_{k=1}^{\infty} b_{n,k} \left( \sum_{j=1}^{\infty} \left( k - 1 \right) \int_0^1 \theta^{j-1} (1 - \theta)^{k-j} d\mu(\theta) x_k \right) \right)^q \right)^{\frac{1}{q}}
$$

$$
\geq \int_0^1 \left( \sum_{n=1}^{\infty} v_n \left( \sum_{k=1}^{\infty} b_{n,k} e_{j,k}(\theta) x_k \right)^q \right)^{\frac{1}{q}} d\mu(\theta)
$$

$$
= \int_0^1 \|E(\theta) x\|_{v,B(q)} d\mu(\theta)
$$

$$
\geq C_B R_B^{\frac{1}{p}} \|x\|_{v,B(q)} \int_0^1 \theta^{-\frac{1}{q}} d\mu(\theta)
$$

$$
\geq C_B R_B^{\frac{1}{p}} \int_0^1 \theta^{-\frac{1}{q}} d\mu(\theta).
$$

Now, consider (i). Let $e_2 = (0, 1, 0, ...)$, then $e_2 \geq 0$ and $\|e_2\|_{v,B(1)} = v_1 b_{2,1} + v_2 b_{2,2} = 1$.

$$
\|H_{\mu} e_2\|_{v,B(1)} = \sum_{n=1}^{\infty} v_n \left( \sum_{k=1}^{\infty} b_{n,k} h_{k,2}(\theta) \right)
$$

$$
\geq \int_0^1 \sum_{n=1}^{\infty} e_{n,2}(\theta) d\mu(\theta)
$$

$$
= \int_{(0,1]} \theta^{-1} d\mu(\theta).
$$

$$
\geq C_B \int_{(0,1]} \theta^{-1} d\mu(\theta).
$$

Hence

$$
L_{v,B(1)}(H_{\mu}) \leq C_B \int_{(0,1]} \theta^{-1} d\mu(\theta).
$$

Combining this with (2.1), we obtain (i). Now, consider (ii). Obviously, (2.1) is an equality if its right-hand side is infinity. For the case that $\mu(\{0\}) + \mu(\{1\}) = 1$, we have

$$
\|H_{\mu} e_2\|_{v,B(q)} = \left( \sum_{n=1}^{\infty} v_n \left( \sum_{k=1}^{\infty} b_{n,k} h_{k,2}(\theta) \right)^q \right)^{\frac{1}{q}}
$$
\[
\left( \sum_{n=2}^{\infty} v_n \left( \sum_{k=1}^{\infty} h_{k,2}(\theta) \right)^q \right)^{\frac{1}{q}} \\
\geq \left( \sum_{n=2}^{\infty} v_n \left( \sum_{k=1}^{\infty} h_{k,2}^q(\theta) \right) \right)^{\frac{1}{q}} \\
\geq \left( \sum_{n=2}^{\infty} v_n h_{n,2}^q(\theta) \right)^{\frac{1}{q}} \\
= \left( \sum_{n=2}^{\infty} v_n \left( \left( n - 1 \right) \int_0^1 \theta (1 - \theta)^{n-2} d\mu(\theta) \right)^q \right)^{\frac{1}{q}} \\
= \mu(\{1\}) = \int_{(0,1]} \theta^{\frac{1}{p}} d\mu(\theta).
\]

this follows that
\[
L_{v,p,q,B}(H_\mu) \leq \int_{(0,1]} \theta^{\frac{1}{p}} d\mu(\theta),
\]
so (2.1) is an equality.

Conversely, let \(0 < q < p \leq 1\), \(B = I\) and assume that \(\mu(\{0\}) + \mu(\{1\}) \neq 1\) and also
\[
\int_{(0,1]} \theta^{-\frac{1}{q}} d\mu(\theta) < \infty,
\]
then \(\mu((0,1)) \neq 0\). Since \(0 < q < 1\), we have
\[
\sum_{n=0}^{\infty} (1 - \theta)^n < \sum_{n=0}^{\infty} (1 - \theta)^{nq}. \quad (\theta \in (0,1))
\]
(2.2)

Applying (2.2), Minkowski’s inequality and monotonicity of \(v\) we have
\[
\int_{(0,1]} \theta^{-\frac{1}{q}} d\mu(\theta) = \int_{(0,1]} \left( \sum_{n=1}^{\infty} (1 - \theta)^n \right)^{\frac{1}{q}} d\mu(\theta) \\
\quad < \int_{(0,1]} \left( \sum_{n=1}^{\infty} (1 - \theta)^{nq} \right)^{\frac{1}{q}} d\mu(\theta) \\
\quad \leq \left\| \left\{ \int_{(0,1]} (1 - \theta)^n d\mu(\theta) \right\} \right\|_q \\
\quad \leq \left\| \left\{ \int_{(0,1]} (1 - \theta)^n d\mu(\theta) \right\} \right\|_{v,q}.
\]
(2.3)

From (2.3) we can find \(0 < \beta < 1\) such that
\[
\int_{(0,1]} \theta^{-\frac{1}{q}} d\mu(\theta) < \beta \left\| \left\{ \int_{(0,1]} (1 - \theta)^n d\mu(\theta) \right\} \right\|_{v,q}.
\]
(2.4)
We claim that
\[
L_{v,p,q,B}(H_\mu) \geq \min \left\{ \beta \frac{q-p}{q} \int_{(0,1]} \theta^{-\frac{1}{q}} d\mu(\theta), \beta \left\| \frac{1}{\sqrt{q}} \left( \int_{(0,1]} (1-\theta)^n d\mu(\theta) \right) \right\|_{v,q} \right\}.
\] (2.5)

Let \( x \geq 0 \), with \( \|x\|_{v,B(p)} = 1 \). We divide the proof into two cases: \( x_{k_0} \geq \beta \) for some \( k_0 \) or \( x_k < \beta \) for all \( k \). For the first case, applying Proposition 2.3, it follows that
\[
\|H_\mu x\|_{v,B(q)} = \left( \sum_{n=1}^{\infty} v_n \left( \sum_{k=1}^{\infty} b_{n,k}(H_\mu(x))_k \right)^q \right)^{\frac{1}{q}}
= \left( \sum_{n=1}^{\infty} v_n \left( H_\mu x \right)_n^{q} \right)^{\frac{1}{q}}
= \left( \sum_{n=1}^{\infty} v_n \left( \sum_{k=1}^{\infty} r_{n,k} x_k \right)^q \right)^{\frac{1}{q}}
\geq x_{k_0} \left( \sum_{n=1}^{\infty} v_n r_{n,k_0}^q \right)^{\frac{1}{q}}
\geq \beta \left\| \frac{1}{\sqrt{q}} \left( \int_{(0,1]} e_{n,k_0}(\theta) d\mu(\theta) \right) \right\|_{v,q}
\geq \beta \left\| \frac{1}{\sqrt{q}} \left( \int_{(0,1]} e_{n,1}(\theta) d\mu(\theta) \right) \right\|_{v,q}
= \beta \left\| \frac{1}{\sqrt{q}} \left( \int_{(0,1]} (1-\theta)^n d\mu(\theta) \right) \right\|_{v,q}.
\]

As for the second case, we have
\[
x_k^q \geq \beta^{q-p} x_k^p,
\]
so
\[
\|x\|_{v,q} = \left( \sum_{k=1}^{\infty} v_k x_k^q \right)^{\frac{1}{q}} \geq \beta^{\frac{q-p}{q}} \left( \sum_{k=1}^{\infty} v_k x_k^p \right)^{\frac{1}{q}} = \beta^{\frac{q-p}{q}}.
\]

Applying (2.1), for the case \( B = I \), we deduce that
\[
\|H_\mu x\|_{v,B(q)} \geq \left( \int_{(0,1]} \theta^{-\frac{1}{q}} d\mu(\theta) \right) \|x\|_{v,B(q)}
= \left( \int_{(0,1]} \theta^{-\frac{1}{q}} d\mu(\theta) \right) \|x\|_{v,q}
= \beta^{\frac{q-p}{q}} \left( \int_{(0,1]} \theta^{-\frac{1}{q}} d\mu(\theta) \right).
\]

Hence, \( \|H_\mu x\|_{v,B(q)} \) is always greater than or equal to the minimum stated at the right-hand side of (2.5). It is clear that \( \beta^{\frac{q-p}{q}} > 1 \). Considering (2.4) and (2.5) together, (ii) is obtained.
\[\square\]
Corollary 2.6. If $F = (F_n)$ is a partition of natural numbers which $N$ is the largest cardinal numbers of $F_n$’s. Then

$$L_{v,p,q,F}(H_{\mu}) \geq N^\frac{1}{p} \int_{[0,1]} \theta^{-\frac{1}{q}}d\mu(\theta) \quad (0 < q \leq p \leq 1).$$

So, Theorem 1.1 is improved.

References