Solvability and numerical method for non-linear Volterra integral equations by using Petryshyn’s fixed point theorem

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Abstract

In this paper, utilizing the technique of Petryshyn’s fixed point theorem in Banach algebra, we analyze the existence of solution for functional integral equations, which includes as special cases many functional integral equations that arise in various branches of non-linear analysis and its application. Finally, we introduce the numerical method formed by modified homotopy perturbation approach to resolving the problem with acceptable accuracy.

Keywords: Fixed point theorem, Measure of non-compactness(MNC), Banach algebra, Functional integral equation(FIE), Modified Homotopy perturbation(MHP)

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FIEs are regarded as a part of the applications of non-linear analysis. It create a very important and significant part of the theories of radiative transfer, mathematical physics, population dynamics, kinetic theory of gases and neutron transport \[1, 4, 10, 11, 12, 21\]. Recently, the concept of MNC is one of the most efficient tools in non-linear analysis to study the solvability of FIEs and differential equations \[11, 16, 17, 24, 25, 27, 33\]. We study a different existence result for the solution of FIEs

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as,
\[
\begin{align*}
    u(s) &= q \left( s, u(\theta(s)), v(s, u(\zeta(s))), \int_0^{\alpha(s)} h(s, r, u(\varphi(r)))dr, \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr \right) \
    u(s) &= q \left( s, v(s, u(\zeta(s))), \int_0^{\alpha(s)} h(s, r, u(\varphi(r)))dr, u(\theta(s)) \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr \right) \
    u(s) &= q \left( s, u(\zeta(s)), u(\theta(s)) \int_0^{\epsilon} h(s, r, u(\varphi(r)))dr, u(\theta(s)) \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr \right)
\end{align*}
\]
(0.1) (0.2) (0.3)

where, \( s \in I_c = [0, c] \). Eq. (1) is the generalization of the some equations which introduced by \([3, 7, 8, 15, 19, 22]\), etc. We apply the concept of MNC and Petryshyn’s fixed point theorem\([29]\), which is generalization of Darbo’s fixed point theorem\([5]\). The main aim is to find the existence result of Eq. (1) and also, work to gain the analytic solution of it by applying the semi-analytic method. Many useful papers have been studied to the existence result for various FIE by Darbo’s fixed point theorem (see\([6, 7, 8, 18, 22, 23]\)). Now, we discuss the principal reason why we examine Eqs. (1, 0.2, 0.3) and what we perform. The first importance is that the conditions given in various papers will be investigated and the second reason is that it joins similar work in this area. The third condition is the bounded condition explains that the ”sublinear condition” that has been recognized in literature will not play a meaningful role here.

This article is divided into 4 sections including the introduction. Section 2, we show some preliminaries and define the concept of MNC. Section 3 is applied to state and prove an existence result for Eq. (1) using Petryshyn’s theorem. In last, we provide some examples that test the utilization of FIE.

1. Preliminaries

In this paper, assume \( \Delta \) be a real Banach space. Let \( B(u, \sigma) \) be a closed ball centre at \( u \) with radius \( \sigma > 0 \). MNC are very powerful tools in non-linear analysis, for example in the operator equations theory and fixed point theory in Banach space.

**Definition 1.1.** [5] The Kuratowski MNC

\[
\gamma(\Omega) = \inf \{ \rho > 0 : \Omega = \bigcup_{j=1}^{m} \Omega_j \text{ with } \text{diam}\Omega_j \leq \rho, \ j = 1, 2, ..., m \}.
\]

**Definition 1.2.** [5] The Hausdorff MNC

\[
\chi(\Omega) = \inf \{ \rho > 0 : \text{there exists a finite } \rho \text{ net for } \Omega \text{ in } \Delta \},
\]
(1.1)

here, from a finite \( \rho \) net for \( \Omega \) in \( \Delta \) it implies, as a set \( \{u_1, u_2, ..., u_m\} \subset \Delta \) such that \( B_\rho(\Delta, u_1), B_\rho(\Delta, u_2), ..., B_\rho(\Delta, u_m) \) over \( \Omega \). These MNC are similar in the sense that

\[
\chi(\Omega) \leq \gamma(\Omega) \leq 2\chi(\Omega), \text{ for any bounded set } \Omega \subset \Delta.
\]

**Theorem 1.3.** Let \( \Omega, Z \subset \Delta \text{ bounded and } \lambda \in \mathbb{R} \). Then

(i) \( \chi(\Omega) = 0 \) iff \( \Omega \) is precompact;

(ii) \( \Omega \subseteq Z \implies \chi(\Omega) \leq \chi(Z) \);

(iii) \( \chi(\partial\Omega) = \chi(\Omega) \);
(iv) \( \chi(\Omega \cup Z) = \max\{\chi(\Omega), \chi(Z)\} \);

(v) \( \chi(\lambda \Omega) = |\lambda| \chi(\Omega) \);

(vi) \( \chi(\Omega + Z) \leq \chi(\Omega) + \chi(Z) \).

Let \( C[0, c] \) be the space of all real valued and continuous function defined on the \([0, c]\) with the standard norm, \( ||u|| = \max\{|u(s)| : s \in [0, c]\} \). \( C[0, c] \) is also the structure of Banach algebra. The modulus of continuity of \( u \) defined as

\[
\omega(u, \rho) = \sup\{|u(s) - u(\eta)| : \eta, s \in [0, c], |s - \eta| \leq \rho\}.
\]

**Theorem 1.4.** [22] The Hausdorff MNC is similar to

\[
\mu(\Omega) = \lim_{\rho \to 0} \sup_{\Omega \subset C[0, c]} \Omega(u, \rho), \text{ for all } \Omega \subset C[0, c].
\]

**Theorem 1.5.** [22] Assume that \( F : \Delta \to \Delta \) is a continuous mapping of \( \Delta \) which holds the condition if for all \( \Omega \subset \Delta \) with \( \Omega \) bounded, \( F(\Omega) \) is bounded and \( \gamma(F\Omega) \leq k\gamma(\Omega), k \in (0, 1) \). If \( \gamma(F\Omega) < \gamma(\Omega) \), for all \( \gamma(\Omega) > 0 \), then \( F \) is called condensing or densifying map.

**Theorem 1.6.** Petryshyn’s [22] Suppose that \( F : B_\sigma \to \Delta \) be a densifying mapping, which fulfill the boundary condition.

If \( F(u) = ku, \text{ for some } u \in \partial B_\sigma \) then \( k \leq 1 \), then the set of fixed points of \( F \) in \( B_\sigma \) is nonempty.

### 2. Main Results

Here, we shall treat Eq. (1) for \( u \) belongs to \( C[0, c] \) under the following assumptions;

\( (T_1) \ q \in C(I_c \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), \nu \in C(I_c \times \mathbb{R}, \mathbb{R}), \) \( h \in C(I_c \times [0, A_1] \times \mathbb{R}, \mathbb{R}), \) \( f \in C(I_c \times [0, A_2] \times \mathbb{R}, \mathbb{R}), \)
and \( \alpha, \beta : I_c \to \mathbb{R}^+, \phi : [0, A_1] \to I_d, \varphi : [0, A_2] \to I_c, \) \( \theta, \zeta : I_c \to I_c, \) are continuous so \( \alpha(s) \leq A_1, \beta(s) \leq A_2 \forall s \in I_c \);

\( (T_2) \ \exists \text{ non-negative constants } d_1 + d_2d_5 < 1, \text{ so} \)

\[
|q(s, v_1, v_2, v_3, v_4) - q(s, x_1, x_2, x_3, x_4)| \leq d_1|v_1 - x_1| + d_2|v_2 - x_2| + d_3|v_3 - x_3| + d_4|v_4 - x_4|;
\]

\[
|\nu(s, v_1) - \nu(s, v_2)| \leq d_5|v_1 - x_1|.
\]

\( (T_3) \ \exists \sigma > 0 \text{ of the inequality} \)

\[
\sup\{|q(s, v_1, v_2, v_3, v_4) : s \in I_c, v_1, v_2 \in [-\sigma, \sigma], v_3 \in [-A_1N_1, A_1N_1],\ v_4 \in [-A_2N_2, A_2N_2]) \leq \sigma,
\]

where,

\[
N_1 = \sup\{|h(s, r_1, u) : \forall s \in I_c, r_1 \in [0, A_1] \text{ and } u \in [-\sigma, \sigma]|,
\]

\[
N_2 = \sup\{|f(s, r_2, u) : \forall s \in I_c, r_2 \in [0, A_2] \text{ and } u \in [-\sigma, \sigma]|.
\]
Theorem 2.1. Under the above hypotheses Eq. \([7]\) has at least one solution in \(\Delta = C(I_c)\).

Proof. Define \(F : B_\sigma \rightarrow \Delta\) as

\[
(Fu)(s) = q \left( \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr, \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr \right),
\]

To prove that \(F\) is continuous on \(B_\sigma\). Takeing \(\rho > 0\) and for any \(u, v \in B_\sigma\) such that \(||u - v|| < \rho\).

We have

\[
| (Fu)(s) - (Fv)(s) | =
q \left( \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr, \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr \right)
- q \left( \int_0^{\alpha(s)} h(s, r, v(\phi(r)))dr, \int_0^{\beta(s)} f(s, r, v(\varphi(r)))dr \right)
\leq
q \left( \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr, \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr \right)
- q \left( \int_0^{\alpha(s)} h(s, r, v(\phi(r)))dr, \int_0^{\beta(s)} f(s, r, v(\varphi(r)))dr \right)
+ q \left( \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr, \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr \right)
- q \left( \int_0^{\alpha(s)} h(s, r, v(\phi(r)))dr, \int_0^{\beta(s)} f(s, r, v(\varphi(r)))dr \right)
+ q \left( \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr, \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr \right)
- q \left( \int_0^{\alpha(s)} h(s, r, v(\phi(r)))dr, \int_0^{\beta(s)} f(s, r, v(\varphi(r)))dr \right)
\leq
d_1|u(\theta(s)) - v(\theta(s))| + d_2|\nu(s, u(\zeta(s)) - \nu(s, v(\zeta(s)))| + d_3 \int_0^{\alpha(s)} |h(s, r, u(\phi(r))) - h(s, r, v(\phi(r)))|dr + d_4 \int_0^{\beta(s)} |f(s, r, u(\varphi(r))) - f(s, r, v(\varphi(r)))|dr
\leq
d_1|u(\theta(s)) - v(\theta(s))| + d_2d_5|u(\zeta(s)) - v(\zeta(s))| + d_3A_1\omega(h, \rho) + d_4A_2\omega(f, \rho)
\leq
(d_1 + d_2d_5)||u - v|| + d_3A_1\omega(h, \rho) + d_4A_2\omega(f, \rho)

where,

\[
\omega(h, \rho) = \sup\{|h(s, r, u) - h(s, r, v)| : s \in I_c, r \in [0, A_1], u, v \in [-\sigma, \sigma], ||u - v|| \leq \rho\},
\]
\[ \omega(f, \rho) = \sup\{|f(s, r, u) - f(s, r, v)| : s \in I, r \in [0, A_2], u, v \in [-\sigma, \sigma], \|u - v\| \leq \rho \}. \]

Since \( h(s, r, u) \) and \( f(s, r, u) \) are uniform continuous on \( I \times [0, A_1] \times \mathbb{R} \) and \( I \times [0, A_2] \times \mathbb{R} \), we conclude that \( \omega(h, \rho) \) and \( \omega(f, \rho) \) as \( \rho \to 0 \). Hence, \( F \) is continuous on \( B_\sigma \). Again, we show that \( F \) satisfies the densifying map. Fixed a arbitrary \( \rho > 0 \) and take \( u \in \Omega \), where \( \Omega \) is bounded subset of \( \Delta \), \( s_1, s_2 \in I \) with \( s_1 \leq s_2 \) such that \( s_1 - s_2 \leq \rho \), we get

\[
\begin{align*}
\|(Fu)(s_2) - (Fu)(s_1)\| &= q \left( s_2, u(\theta(s_2)), \nu(s_2, u(\zeta(s_2))), \int_0^{\alpha(s_2)} h(s_2, r, u(\phi(r)))dr, \int_0^{\beta(s_2)} f(s_2, r, u(\varphi(r)))dr \right) \\
&- q \left( s_1, u(\theta(s_1)), \nu(s_1, u(\zeta(s_1))), \int_0^{\alpha(s_1)} h(s_1, r, u(\phi(r)))dr, \int_0^{\beta(s_1)} f(s_1, r, u(\varphi(r)))dr \right) \\
&\leq q \left( s_2, u(\theta(s_2)), \nu(s_2, u(\zeta(s_2))), \int_0^{\alpha(s_2)} h(s_2, r, u(\phi(r)))dr, \int_0^{\beta(s_2)} f(s_2, r, u(\varphi(r)))dr \right) \\
&- q \left( s_1, u(\theta(s_1)), \nu(s_1, u(\zeta(s_1))), \int_0^{\alpha(s_1)} h(s_1, r, u(\phi(r)))dr, \int_0^{\beta(s_1)} f(s_1, r, u(\varphi(r)))dr \right) \\
&+ q \left( s_2, u(\theta(s_2)), \nu(s_2, u(\zeta(s_2))), \int_0^{\alpha(s_2)} h(s_2, r, u(\phi(r)))dr, \int_0^{\beta(s_1)} f(s_1, r, u(\varphi(r)))dr \right) \\
&- q \left( s_2, u(\theta(s_2)), \nu(s_1, u(\zeta(s_1))), \int_0^{\alpha(s_1)} h(s_1, r, u(\phi(r)))dr, \int_0^{\beta(s_1)} f(s_1, r, u(\varphi(r)))dr \right) \\
&\leq d_1|u(\theta(s_2)) - u(\theta(s_1))| + d_2|\nu(s_2, u(\zeta(s_2)) - \nu(s_1, u(\zeta(s_1)))|
\end{align*}
\]
\[ + \int_{\beta(s_1)}^{\beta(s_2)} |f(s_2, r, u(\varphi(r)))|dr + \omega_q(I_c, \rho). \]

For simplify,

\[ \omega_q(I_c, \rho) = \sup\{|q(s, v_1, v_2, v_3, v_4) - q(\hat{s}, v_1, v_2, v_3, v_4) : |s - \hat{s}| \leq \rho, s, \hat{s} \in I_c, v_1, v_2 \in [-\sigma, \sigma], v_3 \in [-A_1N_1, A_1N_1], v_4 \in [-A_2N_2, A_2N_2]|. \]

\[ \omega_h(I_c, \rho) = \sup\{|h(s, r, u) - h(\hat{s}, r, u) : |s - \hat{s}| \leq \rho, s, \hat{s} \in I_c, r \in [0, A_1], u \in [-\sigma, \sigma]|. \]

\[ \omega_f(I_c, \rho) = \sup\{|f(s, r, u) - f(\hat{s}, r, u) : |s - \hat{s}| \leq \rho, s, \hat{s} \in I_c, r \in [0, A_2], u \in [-\sigma, \sigma]|. \]

\[ \omega_v(I_c, \rho) = \sup\{|\nu(s, v_1) - \nu(\hat{s}, v_1) : |s - \hat{s}| \leq \rho, s, \hat{s} \in I_c, v_1 \in [-\sigma, \sigma]|. \]

\[ \omega(\alpha) = \sup\{|\alpha(s) - \alpha(\hat{s})| : |s - \hat{s}| \leq \rho, s, \hat{s} \in I_c|. \]

From above relation, we get

\[ |(Fu)(s_2) - (Fu)(s_1)| \leq d_1u(\theta(s_2)) - u(\theta(s_1))| + d_2d_5|u(\zeta(s_2)) - u(\zeta(s_1))| + d_2\omega_v(I_c, \rho) + d_3\omega_h(I_c, \rho) + d_3\omega_f(I_c, \rho) + d_4N_2\omega(\beta, \rho) + \omega_q(I_c, \rho). \]

Taking limit as \( \rho \to 0 \), we get \( \omega(Fu, \rho) \leq (d_1 + d_2d_5)\omega(u, \rho) \), this gives the relation \( \chi(F\Omega) \leq (d_1 + d_2d_5)\chi(\omega) \), then \( F \) is a condensing map. Let \( u \in \partial B_\sigma \) and if \( Fu = ku \) then we obtain

\[ ||Fu|| = k||u|| = k\sigma \text{ and by (T_3),} \]

\[ |Fu(s)| = \left| q\left(s, u(\theta(s)), \nu(s, u(\zeta(s)), \int_0^{\alpha(s)} h(s, r, u(\varphi(r)))dr, \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr \right) \right| \leq \sigma \]

\( \forall s \in I_c, \) hence \( ||Fu|| \leq \sigma \) i.e \( k \leq 1 \). \( \square \)

Second, we will study the Eq. \([0.2]\) under the following assumptions,

(B1) \( q \in C(I_c \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \nu \in C(I_c \times \mathbb{R}, h \in C(I_c \times [0, A_1] \times \mathbb{R}, \mathbb{R}), f \in C(I_c \times [0, A_2] \times \mathbb{R}, \mathbb{R}), \) and \( \alpha, \beta : I_c \to \mathbb{R}^+, \phi : [0, A_1] \to I_d, \varphi : [0, A_2] \to I_c, \theta, \zeta : I_c \to I_c, \) are continuous so \( \alpha(s) \leq A_1, \beta(s) \leq A_2 \) \( \forall s \in I_c; \)

(B2) \( \exists \) non-negative constants \( d_1d_4 + d_3A_2N_2 < 1 \) so

\[ |q(s, v_1, v_2, v_3) - q(s, x_1, x_2, x_3) | \leq d_1|v_1 - x_1| + d_2|v_2 - x_2| + d_3|v_3 - x_3|; \]

\[ |\nu(s, v_1) - \nu(s, v_2) | \leq d_4|v_1 - x_1|, \]

(B3) \( \exists \) a \( \sigma > 0 \) of the inequality

\[ \sup\{|q(s, v_1, v_2, v_3) : s \in I_c, v_1 \in [-\sigma, \sigma], v_2 \in [-A_1N_1, A_1N_1], v_3 \in [-A_2N_2, A_2N_2]\} \leq \sigma, \]

where, \( N_1 = \sup\{|h(s, r_1, u) : \forall s \in I_c, r_1 \in [0, A_1] \text{ and } u \in [-\sigma, \sigma]|, \)

\( N_2 = \sup\{|f(s, r_2, u) : \forall s \in I_c, r_2 \in [0, A_2] \text{ and } u \in [-\sigma, \sigma]|. \)

Then Eq. \([0.2]\) has at least one solution in \( I_c. \)

**Proof.** The proof is relevant to the Theorem \([2.1]\) and leave this parts. \( \square \)

Third, we will study the Eq. \([0.3]\) under the following assumptions,
Corollary 2.3. Suppose that

\[(G_1) \quad q \in C(I_c \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), \quad h \in C(I_c \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), \quad f \in C(I_c \times [0, A] \times \mathbb{R}, \mathbb{R}),\]
and \(\beta : I_c \rightarrow \mathbb{R}^+, \varphi : [0, A] \rightarrow I_c, \theta, \zeta : I_c \rightarrow I_c,\) are continuous so \(\beta(s) \leq A \quad \forall \ s \in I_c.\)

\[(G_2) \quad \exists\ \text{non-negative constants} \ d_1 + d_2cN_1 < 1, \ so\]
\[|q(s, v_1, v_2, v_3) - q(s, x_1, x_2, x_3)| \leq d_1|v_1 - x_1| + d_2|v_2 - x_2| + d_3|v_3 - x_3|.
\]

\[(G_3) \quad \exists\ \sigma > 0 \ of \ the \ inequality\]
\[\sup\{|q(s, v_1, v_2, v_3)| : s \in I_c, v_1 \in [-\sigma, \sigma], v_2 \in [-cN_1, cN_1], v_3 \in [-AN_2, AN_2]\} \leq \sigma,
\]
where,
\[N_1 = \sup\{|h(s, r_1, u)| : \forall s, r_1 \in I_c, \ and \ u \in [-\sigma, \sigma]\},\]
\[N_2 = \sup\{|f(s, r_2, u)| : \forall s \in I_c, r_2 \in [0, A] \ and \ u \in [-\sigma, \sigma]\}.
\]

Then Eq.\( (0.3) \) has at least one solution in \( I_c \).

**Proof.** The proof is relevant to the Theorem 2.1 and leave this parts. \( \square \)

**Corollary 2.2.** Suppose that

\[(D_1) \quad q \in C(I_c \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), \quad h \in C(I_c \times [0, A_1] \times \mathbb{R}, \mathbb{R}), \quad and \ \alpha : I_c \rightarrow \mathbb{R}^+, \quad \phi : [0, A_1] \rightarrow I_c, \quad \theta : I_c \rightarrow I_c,\]
are continuous such that \(\alpha(s) \leq A_1 \quad \forall \ s \in I_c;\)

\[(D_2) \quad \text{There exists non-negative constant} \ d \in (0, 1) \ such \ that\]
\[|q(s, v_1, v_2) - q(s, x_1, x_2)| \leq d(|v_1 - x_1| + |v_2 - x_2|);\]
and there exists non-negative constants \(b_1\) such that; \(|q(s, 0, 0)| \leq b_1\)

\[(D_3) \quad \text{There exists constants} \ c_1 \ and \ c_2 \ such \ that; \ |h(s, r, u)| \leq c_1 + c_2|u| ,\]
Moreover \(d + A_1dc_2 < 1.\)

Then the following equation has at least one solution in \( I_c.\)

\[u(s) = q \left( s, u(\theta(s)), \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr \right), \ s \in I_c = [0, c]. \quad (2.1)\]

**Proof.** Let \( \sigma = \frac{F_2}{1 - F_1}, \) where \( F_1 = d + A_1dc_2, F_2 = A_1dc_1 + b_1, \) and \(q(s, v_1, v_2, v_3, v_4) = q(s, v_1, v_3),\)
also \( v_1 = u(\theta(s)) \) and \( v_2 = \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr.\) We see that \( (T_2) \) is led by \( (D_2).\) Again, we show that \( D_3 \) is also fulfill, then
\[|u(s)| = |q \left( s, u(\theta(s)), \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr \right)|,
\[\leq d|u(\theta(s))| + d \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr| + |q(s, 0, 0)|,
\[\leq (d + A_1dc_2)||u|| + A_1dc_1 + b_1, \ for \ all \ s \in I_c,
\]
consequently, \(\sup|q(s, v_1, v_3)| \leq F_1\sigma + F_2 = F_1\frac{F_2}{1 - F_1} + F_2 = \sigma.\)

**Corollary 2.3.** Suppose that
\( (E_1) \) Let \( q \in C(I_c \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), K \in C(I_c \times \mathbb{R}, \mathbb{R}), h \in C(I_c \times [0, A_1] \times \mathbb{R}, \mathbb{R}), f \in C(I_c \times [0, A_2] \times \mathbb{R}, \mathbb{R}), \) and \( \alpha, \beta : I_c \to \mathbb{R}^+, \phi : [0, A_1] \to I_c, \varphi : [0, A_2] \to I_c, \theta : I_c \to I_c, \) are continuous such that \( \alpha(s) \leq A_1, \beta(s) \leq A_2 \) \( \forall s \in I_c. \)

\( (E_2) \) There exists non-negative constants \( d \in (0, 1) \) and \( b_1, b_2 \) such that
\[
|q(s, v_1, v_2) - q(s, x_1, x_2)| \leq d(|v_1 - x_1| + |v_2 - x_2|), \quad |K(s, v_1) - K(s, x_1)| \leq d|v_1 - x_1|;
\]
\[
|K(s, 0)| \leq b_1, \quad |q(s, 0, 0)| \leq b_2.
\]

\( (E_3) \) There exists the constants \( c_1, c_2, c_3 \) and \( c_4 \) such that
\[
|h(s, r, u)| \leq c_1 + c_2|u|, \quad |f(s, r, u)| \leq c_3 + c_4|u|,
\]
Moreover, \( d + c_2A_1d + c_4A_2d < 1. \)

Then the following equation has at least one solution in \( I_c = [0, c]. \)
\[
u(s) = K(s, u(\theta(s))) + q \left( s, \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr, \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr \right), \quad \tag{2.2}
\]

**Proof.** Let \( \sigma = \frac{H_2}{1 - H_1}, \) where \( H_1 = d + c_2A_1d + c_4A_2d, \quad H_2 = b_1 + c_1A_1d + c_3A_2d + b_2, \) and
\[
q(s, v_1, v_2, v_3, v_4) = K(s, v_1) + q(s, v_3, v_4),
\]
where \( v_1 = K(s, u(\theta(t))), v_2 = \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr \) and \( v_3 = \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr. \) We see that \( (T_2) \) is led by \( (E_2). \) Now, we show that \( E_3 \) is also fulfill, then
\[
|u(s)| &= \left| K(s, u(\theta(s))) + q \left( s, \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr, \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr \right) \right|
\]
\[
\leq \left| K(s, u(\theta(s))) - K(s, 0) \right| + \left| K(s, 0) \right| + \left| q(s, \int_0^{\alpha(s)} h(s, r, u(\phi(r)))dr \right| + \left| \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr - q(s, 0, 0) \right|,
\]
\[
\leq |d||u|| + b_1 + A_1d(c_1 + c_2||u||) + A_2d(c_3 + c_4||u||) + b_2,
\]
\[
\leq (d + c_2A_1d + c_4A_2d)||u|| + b_1 + c_1A_1d + c_3A_2d + b_2,
\]
for all \( s \in I_c, \) consequently \( \sup|q(s, v_1, v_2, v_3, v_4)| \leq H_1\sigma + H_2 = H_1\frac{H_2}{1 - H_1} + H_2 = \sigma. \)

**Corollary 2.4.** [19] Let
\( (J_1) \) \( h, r \in C(I \times I \times \mathbb{R}, \mathbb{R}), \quad q \in C(I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), \quad \forall s \in I = [0, 1], \)
\( (J_2) \) There exists constant \( d \in (0, 1) \) such that
\[
|q(s, v_1, v_2, v_3) - q(s, x_1, x_2, x_3)| \leq d(|v_1 - x_1| + |v_2 - x_2| + |v_3 - x_3|);
\]
and there exists non-negative constants \( b_1 \) such that, \( |q(s, 0, 0)| \leq b_1; \)
(J₃) There exists the non-negative constants c₁, c₂, c₃ and c₄ such that

\[ |h(s, r, u)| \leq c₁ + c₂|u|, \quad |f(s, r, u)| \leq c₃ + c₄|u|, \quad \text{and moreover, } d + dc₂ + dc₄ < 1. \]

Then the following equation has at least one solution in \( I = [0, 1] \).

\[ u(s) = q \left( s, u(s), \int_0^1 h(s, r, u(r))dr, \int_0^s f(s, r, u(r))dr \right), s \in I. \]  \( (2.3) \)

**Proof.** Let \( \sigma = \frac{Q₁}{1 - Q₂} \), where \( Q₁ = (d + dc₂ + dc₄) \), \( Q₂ = dc₁ + dc₃ + b₁ \), and \( q(s, v₁, v₂, v₃, v₄) = q(s, v₁, v₂, v₃, v₄) \), in this \( \phi(r) = \varphi(r) = r, \theta(s) = s, \alpha(s) = 1, \beta(s) = s, v₂ = \int_0^1 h(s, r, u(r))dr \) and \( v₃ = \int_0^s f(s, r, u(r))dr \). Thus (T₂) is conducted by (J₂). We show that (J₃) holds

\[ |u(s)| = \left| q \left( s, u(s), \int_0^1 h(s, r, u(r))dr, \int_0^s f(s, r, u(r))dr \right) \right|, \]

\[ \leq d|u(s)| + d \left| \int_0^1 h(s, r, u(r))dr \right| + d \left| \int_0^s f(s, r, u(r))dr \right| + |q(s, 0, 0)|, \]

\[ \leq d|u(s)| + d(c₁ + c₂||u||) + d(c₃ + c₄||u||) + b₁, \]

\[ (d + dc₂ + dc₄)||u|| + dc₁ + dc₃ + b₁, \]

for all \( s \in I = [0, 1] \), consequently \( \sup \{q(s, v₁, v₂, v₃, v₄)\} \leq Q₁\sigma + Q₂ = \sigma. \square \)

**Corollary 2.5.** Putting \( q(s, v₁, v₂, v₃, v₄) = q(s, v₂, v₃, v₄) \) and \( A₁ = A₂ = A \), Eq. \( (1) \) reduces to following FIE studied in \([13]\)

\[ u(s) = q \left( s, \nu(s, \zeta(s)), \int_0^{\alpha(s)} h(s, r, u(\varphi(r)))dr, \int_0^{\beta(s)} f(s, r, u(\varphi(r)))dr \right). \]  \( (2.4) \)

**Corollary 2.6.** Replacing \( q(s, v₁, v₂, v₃, v₄) = q(s, v₁, v₂, v₃) \), \( \nu(s, v₁) = v₁ \) and \( A₂ = A \), Eq. \( (1) \) reduces to following FIE studied in \([22]\)

\[ u(s) = q \left( s, u(\theta(s)), \zeta(s), \int_0^{\alpha(s)} h(s, r, u(\varphi(r)))dr \right). \]  \( (2.5) \)

**Corollary 2.7.** Putting \( q(s, v₁, v₂, v₃, v₄) = \tilde{q}(s, v₁) + v₂v₃ \), and \( A₁ = A \), Eq. \( (1) \) reduces to following FIE studied in \([28]\)

\[ u(s) = \tilde{q}(s, u(\theta(s))) + \nu(s, \zeta(s)) \int_0^{\alpha(s)} h(s, r, u(\varphi(r)))dr. \]  \( (2.6) \)

**Corollary 2.8.** Substituting \( q(s, v₁, v₂, v₃, v₄) = 1 + v₁v₃, \theta(s) = \alpha(s) = \phi(s) = s, \) and \( h(s, r, u) = \frac{s}{s + r}\psi(r)u \). Then Eq. \( (1) \) reduces to Chandrasekhar integral equation in radiative transfer \([11]\).

\[ u(s) = 1 + u(s) \int_0^s \frac{s}{s + r}\psi(r)u(r)dr, s \in I_c = [0, c]. \]  \( (2.7) \)

**Corollary 2.9.** Putting \( q(s, v₁, v₂, v₃, v₄) = a(s) + v₃, \alpha(s) = s \) and \( \phi(r) = r \), Eq. \( (1) \) reduces to Volterra Urysohn integral equation

\[ u(s) = a(s) + \int_0^s h(s, r, u(r))dr. \]  \( (2.8) \)

**Corollary 2.10.** Putting \( q(s, v₁, v₂, v₃, v₄) = b(s) + v₄, \beta(t) = c \) and \( \varphi(r) = r \), Eq. \( (1) \) reduces to Urysohn integral equation

\[ u(s) = b(s) + \int_0^c f(s, r, u(r))dr. \]  \( (2.9) \)
3. An Example

**Example 3.1.** Let the following Volterra non-linear FIE:

\[
\begin{align*}
    u(s) &= \frac{1}{3} \left( \frac{s^2}{1 + s^2} \arctan(|u(\sqrt{s})|) \right) + \frac{1}{3} \int_0^s e^{-2s^2 r \sin(u(r))} dr \\
    &\quad + \frac{1}{3} \int_0^s e^{-3s^2 r} (e^{\sqrt{s}} + s \cos(r) + \frac{1}{2} u(r^2)) dr, \quad s \in [0, 1]
\end{align*}
\]

(3.1)

Put, \( \theta(s) = \sqrt{s}, \alpha(s) = s, \phi(s) = s, \beta(s) = s^2, \varphi(s) = s^2, \forall s \in [0, 1], \)

\[ q(s, v_1, v_2, v_3, v_4) = q_1(s, v_1, v_2) + q_2(s, v_3, v_4), \]

where, \( q_1(s, v_1, v_2) = 0v_1 + \frac{1}{3}v_2, \)

\[ q_2(s, v_3, v_4) = \frac{v_3}{3} + \frac{v_4}{3}, \]

\[ v_2 = \left( \frac{s^2}{1+s^2} \right) \arctan(|u(\sqrt{s})|), \]

\[ v_3 = \int_0^{\alpha(s)} h(s, r, u(\phi(r))) dr, \]

\[ v_4 = \int_0^{\beta(s)} f(s, r, u(\varphi(r))) dr, \]

and

\[ h(s, r, u(\phi(r))) = \frac{e^{-2s^2 r \sin(u(r))}}{4 + |\sin(u(r))|}, f(s, r, u(\varphi(r))) = e^{-3s^2 r} (e^{\sqrt{s}} + s \sin(r) + \frac{1}{2} u(r^2)). \]

It is seen that these functions hold \((T_1)\) and \((T_2)\). Now, we check that \((T_3)\) also holds. Choose \( \sigma = \frac{2}{3} + \frac{2}{3} e \) then \( N_1 \leq \frac{1}{4}, N_2 \leq \frac{4e}{3} + \frac{17}{12} \).

\[ \sup\{q(s, v_1, v_2, v_3, v_4) : s \in [0, 1], v_1, v_2 \in [-\sigma, \sigma], v_3 \in [-AN_1, AN_1], v_4 \in [-AN_2, AN_2]\} \]

\[ \leq \sup\{\frac{1}{3}(u(s) + v_2 + v_3) : s \in [0, 1], -\frac{1}{4} \leq v_2 \leq \frac{1}{4}, -\frac{4e}{3} + \frac{17}{12} \leq v_3 \leq \frac{4e}{3} + \frac{17}{12}\} \leq \frac{5}{6} + \frac{2}{3} e. \]

Hence, from Theorem 2.1 Eq. (3.1) has at least one solution in \( C[0, 1] \). \( \square \)

4. An iterative algorithm to solve Example 4.1

Here, we utilize a sequence of MHP and Adomian decomposition method to solve Eq. (3.1). Homotopy perturbation is a powerful concept in perturbations theory and topology [9, 20]. Any modifications of MHP and Adomian decomposition method can be viewed in [13, 30, 31] and [2, 32] respectively. In this proposed method, we discreet a non-linear functional equation to some smaller difficulties and to free of non-linearity, we apply a linear combination of Adomian polynomials. Therefore we present an iterative algorithm to solve the above problem.

Now, take the general form of Eq. (3.1) as,

\[ M(s, u(s)) - g(s, u(s)) = 0, \quad s \in [0, 1] \]

(4.1)

where \( M \) is a non-linear integral operator and \( g \) is a function. According to [30, 31], divide the operator \( M \) to some nonlinear or linear operators as \( M_1 \) and \( M_2 \). Also \( g \) converts to \( g_1 \) and \( g_2 \). Thus (4.1) can be represented by \( M_1(u) - g_1(s) + M_2(u) - g_2(s, u(s)) = 0 \). Therefore we can define a MHP in this form:

\[ H(\partial, p) = M_1(\partial) - g_1(s) + p(M_2(\partial) - g_2(s, \partial(s))) = 0, \quad p \in [0, 1] \]

(4.2)

\[ u(s) \simeq \partial(s) = \partial_0(s) + p\partial_1(s) + p^2\partial_2(s) + p^3\partial_3(s) + \ldots, \]

(4.3)
here \( p \) is an embedding parameter. Putting parameter \( p = 0 \) to \( p = 1 \) we can get \( M_1(\vartheta) = g_1(s) \) to \( M(\vartheta) = g(s, u(s)) \). In this way, we achieve the solution of (4.1) for \( p = 1 \) and \( u(s) \simeq \lim_{p \to 1} \vartheta(s) \). To introduce operators \( M_1, M_2 \) and functions \( g_1, g_2 \), we consider to the non-linear functional Volterra integral equation (3.1) as follows,

\[
M_1(u(s)) = u(s),
\]

\[
M_2(u(s)) = \int_0^s k_1(s, r) \sin(u(r)) \frac{\sin(u(r))}{4 + |\sin(u(r))|} dr + \int_0^s k_2(s, r)(2e^{\sqrt{\vartheta}} + 2\sqrt{s} \cos(r) + u(r^2)) ds
\]

\[
+ \frac{1}{3}(s^2(\frac{s^2}{1 + s^2})\arctan(|u(\sqrt{s})|)) = 0, \hspace{1em} s \in [0, 1], \tag{4.4}
\]

Then, we have

\[
k_1(s, r) = -\frac{1}{3}re^{-2s^2}, \quad k_2(s, r) = -\frac{1}{6}e^{-3r}.
\]

Substituting (4.5) and (4.3) in (4.2) leads to,

\[
\left( \sum_{i=0}^{\infty} p^i \hat{A}_i(s) \right) - g_1(s) + p \left( \int_0^s k_1(s, r) \frac{\sin(\sum_{i=0}^{\infty} p^i \hat{A}_i(r))}{4 + |\sin(\sum_{i=0}^{\infty} p^i \hat{A}_i(r))|} dr \right)
\]

\[
+ \int_0^s k_2(s, r)(2e^{\sqrt{\vartheta}} + 2\sqrt{s} \cos(r) + \sum_{i=0}^{\infty} p^i \hat{A}_i(r^2)) dr - g_2(s, \sum_{i=0}^{\infty} p^i \hat{A}_i(s)) = 0. \tag{4.6}
\]

For relief, operator \( M_2 \) is converted to operators \( \hat{M}_2 \) and \( \hat{\hat{M}}_2 \) and we use Adomian polynomials for approximate nonlinear terms,

\[
\hat{M}_2(\sum_{i=0}^{\infty} p^i \hat{A}_i(r)) = \frac{\sin(\sum_{i=0}^{\infty} p^i \hat{A}_i(r))}{4 + |\sin(\sum_{i=0}^{\infty} p^i \hat{A}_i(r))|} = \sum_{i=0}^{\infty} p^i \hat{\hat{A}}_i(r)
\]

\[
\hat{\hat{M}}_2(\sum_{i=0}^{\infty} p^i \hat{A}_i(r)) = \sum_{i=0}^{\infty} p^i \hat{A}_i(r^2) = \sum_{i=0}^{\infty} p^i \hat{\hat{\hat{A}}}_i(r)
\]

\[
g_2(s, \sum_{i=0}^{\infty} p^i \hat{A}_i(s)) = \sum_{i=0}^{\infty} p^i \hat{A}_i(s). \quad p \in [0, 1]. \tag{4.7}
\]

In which Adomian polynomials are given as,

\[
\hat{A}_k(r) = \frac{1}{k!} \frac{d^k}{dp^k} \sin(\sum_{i=0}^{\infty} p^i \hat{A}_i(r)) \bigg|_{p=0},
\]

\[
\hat{\hat{A}}_k(r) = \frac{1}{k!} \frac{d^k}{dp^k} \sum_{i=0}^{\infty} p^i \hat{A}_i(r^2)) \bigg|_{p=0},
\]

\[
A_k(s) = \frac{1}{k!} \frac{d^k}{dp^k} g_2(s, \sum_{i=0}^{\infty} p^i \hat{A}_i(s)) \bigg|_{p=0}. \tag{4.8}
\]
We keep some approximations of the solution (3.1) as, from (4.8) Adomian polynomials are,

\[
\vartheta_0(s) + p\vartheta_1(s) + p^2\vartheta_2(s) + \cdots - g_1(s)) + p\left(\int_0^s k_1(s,r)\sum_{i=0}^{\infty} p^i\hat{A}_i(r)dr\right)
\]

\[
\int_0^s k_2(s,r)(2e^{\sqrt{s}} + 2\sqrt{s} \cos(r) + \sum_{i=0}^{\infty} p^i\hat{A}_i(r))dr - \sum_{i=0}^{\infty} p^iA_i(s)) = 0,
\]

with rearranging (4.9) in terms of \(p\) powers and taking of the coefficients of \(p\) powers equal to zero, we approach an iterative algorithm for numerical solution of (3.1).

**Algorithm:**

\[
\vartheta_0(s) = g_1(s),
\]

\[
\vartheta_{k+1}(s) = -\int_0^s k_1(s,r)\hat{A}_k(r)dr - \int_0^s k_2(s,r)(2e^{\sqrt{s}} + 2\sqrt{s} \cos(r) + \hat{A}_k(r))dr
\]

\[
+ A_k(s), \quad k = 0, 1, 2, \cdots
\]

For convergence of these kinds of algorithm see [19]. Since in (3.1), \(u(0) = 0\), and solution space is \(C[0,1]\), then a simple choice for start point in algorithm (4.10) is \(\vartheta_0(s) = g_1(s) = 0\) or \(s\). Therefore from (4.8) Adomian polynomials are,

\[
\hat{A}_0(r) = \frac{\sin(\vartheta_0(r))}{4 + |\sin(\vartheta_0(r))|}, \quad \hat{A}_0(r) = \vartheta_0(r^2), \quad A_0(s) = g_2(s, \vartheta_0(s)).
\]

Thus in algorithm (4.10) we can get

\[
\vartheta_0(s) = g_1(s) = 0
\]

\[
\vartheta_1(s) = -\int_0^s k_1(s,r)\hat{A}_0(r)dr - \int_0^s k_2(s,r)(2e^{\sqrt{s}} + 2\sqrt{s} \cos(r) + \hat{A}_0(r))dr + A_0(s)
\]

\[
= \frac{1}{9} - \frac{1}{9}e^{-3s^2+s} + \frac{\sqrt{s}}{10} - \frac{1}{10}e^{-3s^2}\sqrt{s} \cos(s^2) + \frac{e^{1/2} \sqrt{\pi}}{18\sqrt{3}}(Erf\left(\frac{1}{2\sqrt{3}}\right) - Erf\left(\frac{1 - 6s}{2\sqrt{3}}\right))
\]

\[
+ \frac{1}{30}e^{-3s^2}\sqrt{s} \sin(s^2).
\]

We keep some approximations of the solution (3.1) as,

\[
u(s) = \sum_{i=0}^{\infty} \vartheta_i(s) = \frac{1}{9} - \frac{1}{9}e^{-3s^2+s} + \frac{\sqrt{s}}{10} - \frac{1}{10}e^{-3s^2}\sqrt{s} \cos(s^2)
\]

\[
+ \frac{e^{1/2} \sqrt{\pi}}{18\sqrt{3}}(Erf\left(\frac{1}{2\sqrt{3}}\right) - Erf\left(\frac{1 - 6s}{2\sqrt{3}}\right)) + \frac{1}{30}e^{-3s^2}\sqrt{s} \sin(s^2).
\]

By considering Figure.1, the approximate solution (4.12) is in space \(C[0,1]\). For validity of this numerical result, we replace (4.12) in (3.1) and comparing both sides of it, the absolute errors in some points are given in table 1. It’s axiomatic that by increasing the number of iterations in algorithm (4.10), we can improve the accuracy in the approximation.

5. Conclusions

In the current work, we studied the existence of a solution for functional non-linear Volterra integral equation. For illustrating the efficiency and applicability of our results we gave some corollary and an example respectively. Also, we offer an iterative algorithm to find the solution of the above problem with acceptable accuracy.
Figure 1: $u(s)$: The sum of the first two terms of the series $4.3$

<table>
<thead>
<tr>
<th>$s$</th>
<th>Absolute errors for $u(s)$</th>
</tr>
</thead>
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<tr>
<td>0.0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>$1.7 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.2</td>
<td>$1.2 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.3</td>
<td>$3.9 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.4</td>
<td>$8.1 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.5</td>
<td>$1.3 \times 10^{-2}$</td>
</tr>
<tr>
<td>0.6</td>
<td>$2.0 \times 10^{-2}$</td>
</tr>
<tr>
<td>0.7</td>
<td>$2.6 \times 10^{-2}$</td>
</tr>
<tr>
<td>0.8</td>
<td>$3.3 \times 10^{-2}$</td>
</tr>
<tr>
<td>0.9</td>
<td>$4.0 \times 10^{-2}$</td>
</tr>
<tr>
<td>1.0</td>
<td>$4.6 \times 10^{-2}$</td>
</tr>
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References


