Fixed point on generalized dislocated metric spaces

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Abstract

In the present paper, we introduce new types of convergence of a sequence in left dislocated and right dislocated metric spaces. Also, we generalize the Banach contraction principle in these newly defined generalized metric spaces.

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1. Introduction

Soon after Maurice Fréchet \cite{2} seminal paper on metric spaces researchers have started to generalize extend his idea. Menger \cite{5} was the first to propose probabilistic metric spaces, a generalization of metric spaces. Afterward a generalization pseudometric spaces/dislocated metric spaces of metric spaces was proposed by Hitzler and Seda \cite{4}, Hitzler \cite{3}, Hitzler and Seda \cite{4} and Beg et al. \cite{1} studied generalization of Banach contraction principle in dislocated metric spaces. Their results were applied in the area of programming language semantics.

Following Waszkiewicz \cite{6,7}, let \((X,d)\) be a distance space where \(d\) is a function from \(X\) into \([0,\infty)\). Define the distance topology on \((X,d)\) as follows:

1. Let \(x \in X\) and \(\epsilon > 0\). Then the set \(B_d(x,\epsilon) := \{y \in X : d(x,y) < d(x,x) + \epsilon\}\) is called ball with centre \(x\) and radius \(\epsilon\).

2. \(N_x := \{A \subseteq X : \exists \text{ some } \epsilon > 0 \text{ such that } B_d(x,\epsilon) \subseteq A\}\).

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Theorem 1.3. Let \((X, d)\) be a complete \(d\)-metric space and let \(f : X \rightarrow X\) be a Banach contraction function. Then \(f\) has a unique fixed point.

We use the following lemma due to Ahmed, Zeyada and Hassan [9].

Lemma 1.4. Let \((X, d)\) be a \(ld\)-metric space. If \(f : (X, d) \rightarrow (X, d)\) is a Banach contraction function, then \((f^n(x_0))\) is a Cauchy sequence for each \(x_0 \in X\).
Lemma 1.5. Let \((X, d)\) be a rd-metric space. If \(f : (X, d) \to (X, d)\) is a Banach contraction function, then \((f^n(x_0))\) is a Cauchy sequence for each \(x_0 \in X\).

Theorem 1.6. Let \((X, d)\) be a complete ld-metric space and let \(f : X \to X\) be a Banach contraction function. Then \(f\) has a unique fixed point.

Theorem 1.7. Let \((X, d)\) be a complete rd-metric space and let \(f : X \to X\) be a Banach contraction function. Then \(f\) has a unique fixed point.

2. Definitions in distance spaces

In this section, we introduce definitions needed for our results in a distance space. As it turns out, these notions can be carried over directly from conventional metrics.

Definition 2.1. A sequence \((x_n)\) in a distance space \((X, d)\) is called a Cauchy sequence if \(\forall \epsilon > 0, \exists n_0 \in \mathbb{N}\) such that \(d(x_m, x_n) < \epsilon\) \(\forall m, n \geq n_0\).

Definition 2.2. A sequence \((x_n)\) q-left-converges to \(x\) iff \(\lim_{n \to \infty} d(x_n, x) = d(x, x)\). In this case \(x\) is called a q-left-limit of \((x_n)\).

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Definition 2.4. A distance space \((X, d)\) is called q-left (resp. q-right) complete if every Cauchy sequence is q-left (resp. q-right) convergent.

Definition 2.5. Let \((X, d_1)\) and \((Y, d_2)\) be distance spaces and let \(f : (X, d_1) \to (Y, d_2)\). Then \(f\) is q-left-continuous iff \(\forall x_0 \in X, \forall \epsilon > 0 \exists \delta(\epsilon) > 0\) such that

\[
|d_1(x, x_0) - d_1(x_0, x_0)| < \delta(\epsilon) \Rightarrow |d_2(f(x), f(x_0)) - d_2(f(x_0), f(x_0))| < \epsilon
\]

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\[
|d_1(x_0, x) - d_1(x_0, x_0)| < \delta(\epsilon) \Rightarrow |d_2(f(x_0), f(x)) - d_2(f(x_0), f(x_0))| < \epsilon
\]

Definition 2.7. A function \(f : X \to X\) is called a Banach contraction function if there exists \(0 \leq \lambda < 1\) such that \(d(f(x), f(y)) \leq \lambda d(x, y)\) for all \(x, y \in X\).

Lemma 2.8. Every subsequence of q-left (resp. q-right) convergent sequence to \(x_0\) is a q-left (resp. q-right) convergent to \(x_0\).

Lemma 2.9. Let \((X, d_1)\) and \((Y, d_2)\) be distance spaces. A mapping \(f : (X, d_1) \to (Y, d_2)\) is q-left-continuous iff \(\forall (x_n)\) in \(X\) q-left- \(d_1\)-converges to \(x_0 \in X, (f(x_n))\) in \(Y\) q-left- \(d_2\)-converges to \(f(x_0) \in Y\).
**Proof.** Let $f$ be $q$-left-continuous and $(x_n)$ be a sequence in $X$. Suppose that $(x_n)$ $q$-left-$d_1$-converges to $x_0 \in X$. Let $\epsilon > 0$. Then $\exists \delta(\epsilon) > 0$ such that

$$|d_1(x, x_0) - d_1(x_0, x_0)| < \delta(\epsilon) \Rightarrow |d_2(f(x), f(x_0)) - d_2(f(x_0), f(x_0))| < \epsilon.$$ 

Then $\exists \delta(\epsilon) > 0$ and $\exists n_0 \in N$ such that $\forall n \geq n_0$, $|d_1(x_n, x_0) - d_1(x_0, x_0)| < \delta(\epsilon)$. Thus

$$|d_2(f(x_n), f(x_0)) - d_2(f(x_0), f(x_0))| < \epsilon.$$ 

Hence, $(f(x_n))$ in $Y$ $q$-left-$d_2$-converges to $f(x_0) \in Y$.

Conversely, suppose that $f$ is not $q$-left-continuous. Then $\exists x_0 \in X, \exists \epsilon > 0$ such that $\forall \delta > 0$,

$$|d_1(x, x_0) - d_1(x_0, x_0)| < \delta(\epsilon) \Rightarrow |d_2(f(x), f(x_0)) - d_2(f(x_0), f(x_0))| \geq \epsilon.$$ 

Then the sequence $(x_n)$ $(x_n = x \forall n \in N)$ $q$-left-$d_1$-converges to $x_0$ but $(f(x_n))$ does not $q$-left-$d_2$-converges to $f(x_0)$. □

We state the following lemma without proof:

**Lemma 2.10.** Let $(X, d_1)$ and $(Y, d_2)$ be distance spaces. A mapping $f : (X, d_1) \rightarrow (Y, d_2)$ is $q$-right continuous iff $\forall (x_n)$ in $X$ $q$-right- $d_1$-converges to $x_0 \in X$, $(f(x_n))$ in $Y$ $q$-right-$d_2$-converges to $f(x_0) \in Y$.

3. A generalization of Banach contraction mapping in left-d-metric space

In this section, we give a generalization of the Banach contraction mapping in left d-metric space.

**Definition 3.1.** A left-$d$-metric space $(X, d)$ is called a $q$-left-Hausdorff space iff every left-$q$-convergent sequence $(x_n)$ in $X$ left-$q$-converges to a unique point in $X$.

**Theorem 3.2.** Let $(X, d)$ be a $q$-left-Hausdorff $q$-left-complete ld-metric space and let $f : X \rightarrow X$ be a $q$-left-continuous Banach contraction mapping. Then $f$ has a unique fixed point.

**Proof.** Existence: from Lemma 1.4, $(f_n(x_0))$ is a Cauchy sequence for each $x_0 \in X$. Since $(X, d)$ is $q$-left complete, then $(f^n(x_0))$ $q$-left-converges to a point $x \in X$, say. From the $q$-left-continuity of the mapping $f$ and Lemma 2.9, $(f^{n+1}(x_0))$ $q$-left-converges to $f(x)$. From Lemma 2.8, $(f^{n+1}(x_0))$ $q$-left-converges to $x$. Since $(X, d)$ is a $q$-left-Hausdorff, then $f(x) = x$. □

Uniqueness: suppose that there are two fixed points $x$ and $y$. Then

$$d(x, y) = d(f(x), f(y)) \leq \lambda d(x, y) = (1 - \lambda)d(x, y) \leq 0,$$

$$d(y, x) = d(f(y), f(x)) \leq \lambda d(y, x) = (1 - \lambda)d(y, x) \leq 0.$$ 

Since $(1 - \lambda) > 0$, then we have $d(x, y) = d(y, x) = 0$. Hence, we obtain from (Mii) that $x = y$.

The following counterexample illustrates that there exists a q-left-Hausdorff q-left-complete ld-metric space in which the converse of Proposition 1.1 [8] is not true.

**Counterexample:** Let $X = \{x, y, z\}$. Define $d : X \times X \rightarrow [0, \infty)$ as follows:

$$d(x, y) = d(z, x) = d(z, y) = \frac{1}{8}, d(y, x) = d(x, z) = d(y, z) = \frac{1}{6}, d(x, x) = \frac{1}{7}, d(y, y) = 0, d(z, z) = \frac{1}{4}.$$
(1) One can easily verify that $(X, d)$ is an ld-metric space.

(2) Any sequence $(x_n)$ in $X$ is one of the following forms:

(a) $\exists n_0 \in N$ such that $\forall n \geq n_0, x_n = x$;

(b) $\exists n_0 \in N$ such that $\forall n \geq n_0, x_n = y$;

(c) $\exists n_0 \in N$ such that $\forall n \geq n_0, x_n = z$;

(d) $\forall n \in N$ such that $x_n = x \exists n \in N$ such that $m > n$ and $x_m = z$ and $\forall k \in N$ such that $x_k = z \exists l \in N$ such that $l > k$ and $x_l = x$;

(e) $\forall n \in N$ such that $x_n = y \exists n \in N$ such that $m > n$ and $x_m = z$ and $\forall k \in N$ such that $x_k = z \exists l \in N$ such that $l > k$ and $x_l = y$;

(f) $\forall n \in N$ such that $x_n = x \exists n \in N$ such that $m > n$ and $x_m = x$ and $\forall k \in N$ such that $x_k = x \exists k \in N$ such that $l > k$ and $x_l = y$.

Since only any sequence of form (a) is a Cauchy sequence and q-left-converges to $x$, then $(X, d)$ is q-left-complete.

(3) One can deduce that any sequence of from (a) which are the only q-left-convergent sequences in $X$, q-left-converges to the unique point $x$. Hence $(X, d)$ is q-left-Hausdorff.

(4) One can verifies that $\tau_d = \{X, \emptyset, \{y\}, \{x, y\}\}$ and note that any sequence of the form (b) $\tau_d$-converges to $x$ but does not q-left-converges to $x$.

**Remark 3.3.** Note that although $(X, d)$ in Counterexample 3.1 is q-left-Hausdorff but $(X, \tau_d)$ is not Hausdorff.

### 4. A generalization of Banach contraction mapping in right-d-metric space

We give a generalization of the Banach contraction mapping in rd-metric space.

**Definition 4.1.** A right-d-metric space $(X, d)$ is called a q-right-Hausdorff space iff every right-q-convergent sequence $(x_n)$ in $X$ right-q-converges to a unique point in $X$.

**Theorem 4.2.** Let $(X, d)$ be a q-left-Hausdorff q-right-complete rd-metric space and let $f : X \to X$ be a q-right-continuous Banach contraction mapping. Then $f$ has a unique fixed point.

**Proof.** Existence: from Lemma 1.2, $(f^n(x_0))$ is a Cauchy sequence for each $x_0 \in X$. Since $(X, d)$ is q-right complete, then $(f^n(x_0))$ q-right-converges to a point $x \in X$, say. From the q-right-continuity of the mapping $f$ and Lemma 2.2, $(f^{n+1}(x_0))$ q-right-converges to $f(x)$. From Lemma 2.1, $(f^{n+1}(x_0))$ q-right-converges to $x$. Since $(X, d)$ is a q-left-Hausdorff, then $f(x) = x$. Uniqueness: suppose that there are two fixed points $x$ and $y$. Then

$$d(x, y) = d(f(x), f(y)) \leq \lambda d(x, y) = (1-\lambda)d(x, y) \leq 0,$$

$$d(y, x) = d(f(y), f(x)) \leq \lambda d(y, x) = (1-\lambda)d(y, x) \leq 0.$$

Since $(1-\lambda) > 0$, then we have $d(x, y) = d(y, x) = 0$. Hence we obtain from $(Mii)$ that $x = y$. The following counterexample illustrate that there exists a q-left Hausdorff q-right-complete rd-metric space in which the converse of Proposition 1.1 is not true. □

**Counterexample:** Let $X = \{x, y, z\}$. Define $d_1 : X \times X \to [0, \infty)$ by $d_1(a, b) = d(b, a) \forall a, b \in X$, where $d$ is defined as in Counterexample 3.1. One can verifies that $(X, d)$ is a q-right-Hausdorff q-right-complete rd-metric space. One can verifies that $\tau_d^{-1} = \{X, \emptyset, \{y\}, \{x, y\}\}$. Note that any sequence of the form (c) $\tau_d^{-1}$-converges to $x$ but does not q-right-converge to $x$. 

Remark 4.3. Note that although $(X, d_1)$ in Counterexample 4.1 is $q$-right-Hausdorff but $(X, \tau_{d_1})$ is not Hausdorff.

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References