Study connection between the Laurent series and residues on the $A(z)$ analytic functions

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Abstract

In this paper, we obtain a formula for residues and prove Laurent expansion and expansion to Taylor series for $A(z)$-analytic functions.

Keywords: $A$-analytic function, Laurent expansion, Residues Computed.

1. Introduction

Let $A(z)$ be an antianalytic function, i.e., $dA/\partial z = 0$ in the domain $D \subset C$; moreover, let $|A(z)| \leq c < 1$ for all $z \in D, c$ is constant. The function $f(z)$ is said to be $A(z)$-analytic in the domain $D$ if for any $z \in D$, the following equality holds:

$$\frac{\partial f}{\partial z} = A(z) \frac{\partial f}{\partial z}. \quad (1.1)$$

We denote by $O_A(D)$ the class of all $A(z)$-analytic functions defined in the domain $D$. Since the antianalytic function is infinitely smooth, $O_A(D)C^\infty(D)$ (see [8]).

We will now study the behavior of $f(z)$ at an isolated singularity $z_0$ by expanding (sound familiar) This series will not in general be a Taylor series.

$$a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots$$

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because Taylor series yield analytic functions, where as \( f(z) \) is not analytic at a pole or essential singularity. The series we will obtain will involve negative (as well as positive) powers of \( z - z_0 \). A series consisting of negative powers looks:

\[
b_0 + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_k}{(z - z_0)^k} + \cdots, \quad k \in \mathbb{N}.
\]

**Theorem 1.1 (Analogue to the Cauchy Theorem).** If \( f \in O_A(D) \cap C(\overline{D}) \),

\[
\int_{\partial D} f(z)(dz + A(z)d\overline{z}) = 0. \tag{1.2}
\]

**Theorem 1.2 (Laurent’s growth).** Let \( f(z) \) be \( A(z) \)– analytic in the ring of lemniscate: \( f(z) \in O_A(L(a, R) \setminus L(a, r)) \), \( r < R \). Then \( f(z) \) will be expanded to the Laurent series in the ring \( r < p < R \):

\[
f(z) = \sum_{k=-\infty}^{\infty} c_{j\psi^k(z, a)} \tag{1.3}
\]

where the coefficients of the series are determined by the formula

\[
c_k = \frac{1}{2\pi i} \int_{\partial L(a, p)} \frac{f(\xi)}{[\psi(\xi, a)]^{k+1}} (d\xi + A(\xi) d\xi), \quad k = 0, \pm 1, \pm 2, \cdots
\]

The series \( (1.3) \) converges uniformly inside of the ring

\[
L(a, R) \setminus L(a, r) = \{ z \in D : r < |\Psi(z, a)| < R \}.
\]

**Example 1.3.** Find the Laurent expansion of the function’s two nonzero terms. \( f(z) = \tan z \) about \( z = \frac{\pi}{2} \).

Let us call \( z = \frac{\pi}{2} + u \).

**Solution.**

\[
f(z) = \frac{\sin(\frac{\pi}{2} + u)}{\cos(\frac{\pi}{2} + u)} = -\frac{\cos u}{\sin u}
\]

by using \( \sin(A + B) = \sin A \cos B + \cos A \sin B \) and \( \cos(A + B) = \cos A \cos B - \sin A \sin B \)

This can be expanded using the Taylor series for \( \sin u \) and \( \cos u \) Where

\[
\sin u = \sum_{j=0}^{\infty} \frac{(-1)^j u^{2j+1}}{(2j + 1)!}
\]

\[
\cos u = \sum_{j=0}^{\infty} \frac{(-1)^j u^{2j}}{(2j)!}
\]

\[
f(z) = -\frac{(1 - \frac{u^2}{2!} + \cdots)}{(u - \frac{u^3}{3!} + \cdots)} = -\frac{1}{u} \left( 1 - \frac{u^2}{2!} + \cdots \right)
\]

The numerator can be increased by using \( \sum_{j=0}^{\infty} u^j = \frac{1}{1-u} \), for \( |u| < 1 \). To obtain, for the first two nonzero terms

\[
f(z) = -\frac{1}{u} \left( 1 - \frac{u^2}{2!} + \cdots \right) (u + \frac{u^2}{3!} + \cdots)
\]

\[
f(z) = -\frac{1}{u} \left( 1 - \frac{u^2}{3} + \cdots \right) = -\frac{1}{(z - \frac{\pi}{2})} + \frac{(z - \frac{\pi}{2})}{3} + \cdots
\]
Example 1.4. \( g(z) = \frac{1}{(z+i)} \) convergent in a perforated disc around the pole in the Laurent series \( z_0 = i \).

**Solution.** We note first that \( g(z) = \frac{1}{(z-i)(z+i)} \). We wish to expand this in positive and negative powers of \( z-i \). It makes sense to expand the factor \( \frac{1}{(z+i)} \) in powers of \( z-i \) and then multiply this expansion by \( \frac{1}{z-i} \) to get the expansion for \( g(z) \).

Usually, we alter the geometric series for \( \frac{1}{1-\tau} \) with a shrewdly chosen \( \tau \). Involving \( -i \). We observe that

\[
\frac{1}{z+i} = \frac{1}{2i} \frac{1}{1+(z-i)} = \frac{-i}{2i} \frac{1}{1-(\frac{i}{2})(z-i)} = \frac{-i}{2i} (1 + \frac{i}{2}(z-i) - \frac{1}{2^2}(z-i)^2 - \frac{i}{2^3}(z-i)^3 + \cdots)
\]

It follows that

\[
g(z) = \frac{1}{z-i} \frac{1}{z+i} = -\sum_{n=0}^{\infty} \left( \frac{i}{2} \right)^{n+1} (z-i)^{n+1}.
\]

2. Residues of \( A(z) \) – analytic function

Let \( f(z) \) be an \( A(z) \) – analytic function in \( D \setminus \{a_1, a_2, \cdots, a_n\} \) and continuous on \( \partial D \), where \( a_1, a_2, \cdots, a_n \) are isolated singular points. Then there exists a number \( r > 0 \) such that

\[
L(a_k, r) \cap L(a_1, r) = \emptyset \text{ for } K \neq 1.
\]

Assume the following relationships are true:

\[
G_r = \{ z \in D : |z - \xi| > r \text{ for all } \xi \in \partial D \}; \quad \bigcup_{k=1}^{n} L(a_k, r) \subset G_r
\]

Where \( \partial G_r \) is an arbitrary piecewise – smooth closed contour lying in the domain \( D \), and containing the points \( a_1, a_2, \cdots, a_n \) inside. Since the function \( f(z) \) is \( A(z) \) – analytic at each point of the closed domain bounded by the contour \( \partial G_r \cup \sum_{k=1}^{n} \partial L(a_k, r) \), then by the Cauchy theorem we have

\[
\oint_{\partial G_r} f(\xi) \omega(z) = \sum_{k=1}^{n} \oint_{\partial L(a_k, r)} f(\xi) \omega(z) \tag{2.1}
\]

where \( (z) = dz + A(z) d\bar{z} \).

**Definition 2.1.** The residue of an \( A(z) \) – analytic function \( f(z) \) at a point \( a \) is the value of the integral of the function \( f(z) \) taken over a sufficiently small \( A(z) \) – lemniscate \( L(a, r) \), divided by \( 2\pi i : \text{res}_{z=a} f(z) = \frac{1}{2\pi i} \oint_{L(a, r)} f(\xi) \omega(z) \).

**Theorem 2.2 (Analogue to the Cauchy residue theorem).** Let \( A(z) \) be analytic everywhere in a domain for a function \( f(z) \). \( G \subset \subset D \) except for an isolated set of singular points and let its boundary \( \partial G \) do not contain singular points. Then \( \oint_{\partial G} f(\xi) \omega(z) = 2\pi i \sum_{k=1}^{n} \text{res}_{z=a_k} z = a_k f(z) \).
Proof. The proof of this theorem follows from the formula (1.2) and Definition 2.1.

Example 2.3. We fix $\xi \in D$ and consider the kernel $K_n(\xi, z) = \frac{n!}{2\pi i} \frac{1}{\psi(\xi, z)^n}$. Then

$$\text{res}_z K_n(\xi, z) = \begin{cases} 0, & n \neq 1, \\ 1, & n = 0. \end{cases}$$

(2.2)

Assume that at the point $z = a$, the function $f(z)$ can be expanded in a Laurent series:

$$f(z) = \sum_{k=-\infty}^{\infty} C_k \left( z - a + \int_{\gamma(a, z)} A(\tau) d\tau \right)^k$$

(2.3)

Theorem 2.4. In an isolated singular point, the residue of an $A(z)$-analytic function $f(z)$, $a \in \mathbb{C}$ is equal to the coefficient $c_{-1}$ of the minus first degree of $\psi(z, a)$ in its Laurent expansion in a neighborhood of the $A(z)$-lemniscat $L(a, r)$ at the point $a$:

$$\text{res}_z f(z) = c_{-1}.$$ (2.4)

Proof. Equality (2.3) is obtained from Eq. (2.4) by integration over a lemniscat $\partial L(a, r)$ using (2.2):

$$\text{res}_z f(z) = \frac{1}{2\pi i} \oint_{\partial L(a, r)} \sum_{k=-\infty}^{\infty} C_k \left( z - a + \int_{\gamma(a, z)} A(\tau) d\tau \right)^k \omega(\xi)$$

$$= \frac{1}{2\pi i} \sum_{k=-\infty}^{\infty} C_k \oint_{\partial L(a, r)} \left( z - a + \int_{\gamma(a, z)} A(\tau) d\tau \right)^k \omega(\xi)$$

$$= \frac{1}{2\pi i} 2\pi i c_{-1} = c_{-1}.$$ \qed

Definition 2.5. A point $z = a$ of an $A(z)$-analytic function $f(z)$ of order $n$ is referred to as a zero if

$$f(z) = \left( z - a + \int_{\gamma(a, z)} A(\tau) d\tau \right)^n g(z),$$

where $g(a) \neq 0$ and $g(z) \in O_A(D)$.

Theorem 2.6. If the $A(z)$-analytic function $f(z)$ has a point $a$ that is not identically equal to zero in any neighborhood of $L(a, r)$, then there exists a natural number $n$ such that $f(z) = \left( z - a + \int_{\gamma(a, z)} A(\tau) d\tau \right)^n \phi(z)$, where the function $\phi(z)$ is $A(z)$-analytic at the point $a$ and is nonzero in some neighborhood of this point.

Remark 2.7. An isolated singular point $a \in \mathbb{C}$ of the function $f(z)$ is removable if and only if the Laurent expansion of $f(z)$ in a neighborhood of $a$ does not contain the principal part, i.e. $f(z) = \sum_{k=0}^{\infty} C_k \left( z - a + \int_{\gamma(a, z)} A(\tau) d\tau \right)^k$.

Definition 2.8. A point $z = a$ is called a pole of an $A(z)$-analytic function $f(z)$ of order $n$ if the point $a$ is a zero of the function $\frac{1}{f(z)}$ of order $n$. 

Theorem 2.9. A pole is an isolated singular point \( a \in \mathbb{C} \) of the \( A(z) \) – analytic function \( f(z) \) if and only if the primary component of the Laurent expansion of the \( A(z) \) – analytic function \( f(z) \) in the vicinity of the point \( a \) contains only a finite (and positive) number of nonzero terms. i.e.

\[
    f(z) = \sum_{k=-n}^{\infty} C_k \left( z - a + \int_{\gamma(a,z)} A(\tau)d\tau \right)^k, \quad n \geq 1.
\]

Proof. 

\( \Rightarrow \) Let \( a \) be pole; since \( \lim_{z \to a} f(z) = \infty \), there exists a punctured neighborhood of the point \( a \) where \( f(z) \) is \( A(z) \) – analytic and nonzero. In this neighborhood the function \( g(z) = \frac{1}{f(z)} \) is \( A(z) \) – function analytic and there exists the \( \lim_{z \to a} g(z) = 0 \). Therefore, \( a \) is a removable point (zero) of the function \( g(z) \) and in the neighborhood \( L(a, r) \) the following expansion holds:

\[
g(z) = \sum_{k=n}^{\infty} b_k \left( z - a + \int_{\gamma(a,z)} A(\tau)d\tau \right)^k.
\]

Then in the same neighborhood we obtain the identity

\[
f(z) = \frac{1}{g(z)} = \frac{1}{\left( z - a + \int_{\gamma(a,z)} A(\tau)d\tau \right)^n} \cdot \frac{1}{b_n + b_{n+1} \left( z - a + \int_{\gamma(a,z)} A(\tau)d\tau \right) + \cdots}
\]

The second factor is \( A(z) \) – analytic function at the point \( a \), and hence it admits a Taylor expansion, we obtain

\[
f(z) = \sum_{k=-n}^{\infty} C_k \left( z - a + \int_{\gamma(a,z)} A(\tau)d\tau \right)^k.
\]

This is the Laurent expansion of \( f(z) \) in the neighborhood \( L(a, r) \) \( \{a\} \) of the point \( a \), and we see that its principal part contains a finite number of terms.

\( \Leftarrow \) Let in a the neighborhood \( (a, r) \) \( \{a\} \), \( f(z) \) be represented by the Laurent expansion whose principal part contains a finite number of terms and let \( c_n \neq 0 \).

Then the function \( f(z) \) and \( g(z) = \psi(z, a)^n \cdot f(z) \) are \( A(z) \) – analytic in this neighborhood. The function \( g(z) \) in the neighborhood considered can represented as follows:

\[
g(z) = c_{-n} + c_{-n+1} \left( z - a + \int_{\gamma(a,z)} A(\tau)d\tau \right) + c_{-n+2} \left( z - a + \int_{\gamma(a,z)} A(\tau)d\tau \right)^2 + \cdots
\]

This equality shows that \( a \) is a removable point and there exists

\[
    \lim_{z \to a} g(z) = c_{-n} \neq 0.
\]

Then the function \( f(z) = \frac{g(z)}{\psi(z, a)^n} \) tends to infinity as \( z \to a \), i.e., \( a \) is a pole. The theorem is proved.

\[\square\]

Definition 2.10. If there is a punctured neighborhood of the lemniscate of a point \( a \in \mathbb{C} \), it is called an isolated singular point of the function \( f(z) \), i.e. (the set \( \theta < |\psi(z, a)| < \gamma \)) if the point \( a \) is finite, or a set \( R < \int_{\theta}^\gamma A(\tau)d\tau < \infty \), \( A \equiv \text{const} \), \(|A| < 1\) if \( a = \infty \) in which the function \( f(z) \) is \( A(z) \) – analytic.

Definition 2.11. An isolated singular point \( a \) of a function \( f(z) \) is called:

(a) a pole if \( \lim_{z \to a} f(z) = \infty \);
(b) An essential singularity if the limit of \( f(z) \) as \( z \to a \) does not exist.
3. What is the formula for calculating residues?

We illustrate some methods by examples.

First method Use the Laurent Expansion.

Example 3.1. Evaluate $I = \int_{C_0} e^{\frac{1}{z}} dz$ Where $C_0$ is the unit circle $|z| = 1$

Solution. The function $f(z) = e^{\frac{1}{z}}$ is analytic for at $z \neq 0$ inside $C_0$ and has the Following Laurent expansion about $z = 0$

$$e^{\frac{1}{z}} = (1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots)$$

So that $\text{res}(f;0) = \frac{1}{z}$. Of course $z_0 = 0$ is the only isolated singularity of $e^{\frac{1}{z}}$ in $C$.

By residue theorem $I = 2\pi i$.

Second method Simple poles

A pole $z_0$ of $f(z)$ is said to be simple if its order is 1, that is, if $f(z)$ may be expressed as $f(z) = \frac{c_{-1}}{z-z_0} + c_0 + c_1 (z - z_0) + \cdots$

$$\text{res}_a f(z) = \lim_{z \to a} [f(z)(z - a + \int_{\gamma(a,z)} A(\tau) d\tau)] \quad (3.1)$$

Example 3.2. Evaluate: $\int_0^{2\pi} \frac{2\cos 2\theta}{5 + 4 \cos \theta} \, d\theta$.

Solution. Let

$$I = \int_0^{2\pi} \frac{2\cos 2\theta}{5 + 4 \cos \theta} \, d\theta = \frac{1}{2} \int_0^{2\pi} \frac{e^{2\theta} + e^{-2\theta}}{5 + 2(e^{i\theta} + e^{-i\theta})} \, d\theta$$

write

$$z = e^{i\theta}, \quad d\theta = \frac{dz}{iz}$$

$$= \frac{1}{2i} \int_C \frac{(z^2 + \frac{1}{z})}{5 + 2(z + \frac{1}{z})} \frac{dz}{iz}$$

$$= \frac{1}{2i} \int_C \frac{z^2 + 1}{z^2(2z^2 + 5z + 2)} \frac{dz}{iz}$$

Where $C$ denotes the unit circle $|z| = 1$, the pole of $f(z)$ is $z^2 (2z + 1)(z + 2) = 0 \implies z = 0, z = -\frac{1}{2}, z = -2$

The poles within the contour $C$ are a simple pole at $z = -\frac{1}{2}$, and a pole of order two at $z = 0$
Now, Residue at \( z = -\frac{1}{2} \) is by (3.1)
\[
\lim_{z \to \frac{1}{2}} \left( z + \frac{1}{2} \right) \frac{1}{2i} \frac{z^4 + 1}{z^2 (2z + 1) (z + 2)}
\]
\[
= \frac{1}{2} \cdot \frac{1}{2i} \frac{\left( -\frac{1}{2} \right)^4 + 1}{\left( -\frac{1}{2} \right)^2 (-\frac{1}{2} + 2)}
\]
\[
= \frac{1}{4i} \cdot \frac{1}{\frac{1}{4} + 1} = \frac{17}{24i}
\]

And residue at \( z = 0 \) is coefficient of \( \frac{1}{z} \) in \( \frac{z^4 + 1}{2i z^2 (2z + 1) (z + 2)} \), where \( z \) is small

Now,
\[
\frac{z^4 + 1}{2i z^2 (2z + 1) (z + 2)} = \frac{1}{4i} \left( 1 + \frac{1}{z^4} \right) \left( 1 + \frac{1}{2z} \right)^{-1} \left( 1 + \frac{2}{z} \right)^{-1}
\]
\[
= \frac{1}{4i} \left( 1 + \frac{1}{z^4} \right) \left( 1 - \frac{1}{2z} + \cdots \right) \left( 1 - \frac{2}{z} + \cdots \right)
\]

The coefficient of \( \frac{1}{z} \) is easily seen to be \( \frac{1}{4i} \left( \frac{-5}{2} \right) \), i.e., \( \frac{-5}{8i} \). Hence by Cauchy’s residue theorem
\[
I = 2\pi i \sum_{k=1}^{\infty} \text{res}_A = 2\pi i \left\{ \frac{17}{24i} + \left( \frac{-5}{8i} \right) \right\} = \pi \frac{5}{6}
\]

**Theorem 3.3.** A \( n \)-th-order pole of an \( A(z) \)-analytic function \( f(z) \) is a point \( z=a \). The following formula applies in this case:
\[
\text{res}_A f(z) = \frac{1}{(n-1)!} \lim_{z \to a} \frac{\partial^{n-1}}{\partial z^{n-1}} \left[ f(z) \left( z - a + \int_{\gamma(a,z)} A(\tau)d\tau \right)^n \right]. \tag{3.2}
\]

**Proof.** Due to Theorem 2.9, an \( A(z) \)-analytic function \( f(z) \) has the form
\[
f(z) = \sum_{k=-n}^{\infty} C_k \left( z - a + \int_{\gamma(a,z)} A(\tau)d\tau \right)^k.
\]

Multiplying both sides of this equation by \( \left( z - a + \int_{\gamma(a,z)} A(\tau)d\tau \right)^n \), we obtain
\[
f(z) \left( z - a + \int_{\gamma(a,z)} A(\tau)d\tau \right)^n = c_{-n} + c_{-n+1} \psi(z, a) + \cdots + c_{-1} \psi(z, a)^n + \psi(z, a)^n h(z). \tag{3.3}
\]

Here \( h(z) = \sum_{k=0}^{\infty} c_k \psi(z, a)^k \). We take the partial derivative of the function \( \psi(z, a) \)
\[
\frac{\partial \psi^k}{\partial z} = k \psi^{k-1} \frac{\partial \psi^k}{\partial z} = k \psi^{k-1}. \tag{3.4}
\]

Using this equation, from (3.3) we obtain
\[
\frac{\partial^{n-1}}{\partial z^{n-1}} \left[ f(z) \left( z - a + \int_{\gamma(a,z)} A(\tau)d\tau \right)^n \right] = (n-1)! \ c_{-1} + \psi(z, a)^n h_1(z), \tag{3.5}
\]
\[
h_1(z) = \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!} c_k \psi(z, a)^k.
\]
Passing to the limit as $z \to a$ in Eq. (3.5), we obtain

$$\text{res}_{z=a} f(z) = \frac{1}{(n-1)!} \lim_{z \to a} \frac{\partial^{n-1}}{\partial z^{n-1}} [f(z) \left( z - a + \int_{\gamma(a,z)} A(\tau) d\tau \right)^n].$$

\[\square\]

**Example 3.4.** Evaluate: $\int_0^{2\pi} e^{-\cos \theta} \cos (n\theta + \sin \theta) d\theta$. When $n$ is a positive integer.

**Solution.** Consider the integral

$$I = \int_0^{2\pi} e^{-\cos \theta} \left[ \cos (n\theta + \sin \theta) - i \sin (n\theta + \sin \theta) \right] d\theta$$

$$= \int_0^{2\pi} e^{-\cos \theta} e^{-(n\theta + \sin \theta)} d\theta$$

$$= \int_0^{2\pi} e^{-(\cos \theta + \sin \theta)} e^{-in\theta} d\theta$$

$$= \int_0^{2\pi} e^{-i\theta} e^{-in\theta} d\theta$$

$$= \int_C \left( e^{-z} \cdot \frac{1}{iz} \right) dz$$

Writing $e^{i\theta} = z$, $d\theta = \frac{dz}{iz}$. Where $C$ denotes the unit circle $|z| = 1$.

$$= \frac{1}{i} \int_C \frac{e^{-z}}{z^{n+1}} dz = \int_C f(z) dz,$$

where $f(z) = \frac{e^{-z}}{z^{n+1}}$

$$= 2\pi i \sum_{k=1}^n \text{res}_A \quad (\text{By Cauchy’s residue theorem}).$$

Obviously the only pole of $f(z)$ within the contour $C$ is $z = 0$ of order $n+1$. At $z = 0$, the residue

$$= \frac{1}{n!} \left[ \frac{d^n}{dz^n} \left( \frac{e^{-z}}{z^n} \right) \right]_{z=0} = \frac{(-1)^n}{n!} = \sum_{k=1}^n \text{res}_{A}^+$$

\[.\] $I = 2\pi i \bullet \frac{(-1)^n}{n!} = \frac{2\pi}{n!} (-1)^n$ i.e.

$$\int_0^{2\pi} e^{-\cos \theta} [\cos (n\theta + \sin \theta) - i \sin (n\theta + \sin \theta)] d\theta = \frac{2\pi}{n!} (-1)^n.$$

Equating real parts, we have

$$\int_0^{2\pi} e^{-\cos \theta} \cos (n\theta + \sin \theta) d\theta = \frac{2\pi}{n!} (-1)^n.$$

4. Conclusions

1. An isolated singular point $a \in \mathbb{C}$ of an $A(z)$-analytic function $f(z)$ is a pole if and only if the principal part of Laurent expansion of the $A(z)$-analytic function $f(z)$ in the neighborhood of the point $a$ contains only a finite (and positive) number of nonzero terms, i.e.

$$f(z) = \sum_{k=0}^\infty c_k \left( z - a + \int_{\gamma(a,z)} A(\tau) d\tau \right)^k.$$
2. An isolated singular point \( a \in \mathbb{C} \) of an \( A(z) \)-analytic function \( f(z) \) is removable if and only if the Laurent expansion of \( f(z) \) in a neighborhood of \( a \) does not contain the principal part, i.e.

\[
f(z) = \sum_{k=0}^{\infty} c_k \left( z - a + \int_{\gamma(a,z)} \frac{A(\tau)d\tau}{(z-\tau)} \right)^k.
\]

3. Prove an analogue to the Cauchy residue theorem.

4. Study \( A(z) \)-analytical functions in one particular case more often, when the function \( A(z) \) is an antianalytic function in the considered domain.

References


